KEY

TO

TREATISE ON ALGEBRA.

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BY

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PREFACE.

In complying at length with a request often made to him, by issuing a Key to his Treatise on Algebra, the Author hopes that he will render that work more useful and acceptable. The time of the public teacher will be saved, as he will be able to discover more readily where a mistake may have been committed in the conduct of an investigation; while important aid will be rendered to those who study the subject without the aid of a teacher. To nearly three hundred of the easiest questions in the Elementary Rules, Fractions, and Involution, answers are not given in the Algebra, in order that they may serve as Class Exercises to be prescribed by the teacher; and many other questions are of such a nature that no answers could be given to them, a proof or the steps of a lengthened development being required. The answers or solutions of all such questions will be found here given, and the Key to correspond throughout with the Fourth and stereotyped edition of the work, which was carefully revised

in every part. An Appendix is subjoined, for which a few additional questions in Progressional Series have been kindly supplied, at the Author's request, by his friend Mr. Munn, of the Edinburgh High School, well known as a most able Mathematician and successful Teacher.

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CONTENTS.

KEY TO ALGEBRA.

I. PRAXIS, p. 13.

1. $5+4-6=3$. 2. $7+2-8=1$, &c. 3. $12-12+5=5$; $80+72-30=122$; $360+432+32+75=899$, &c. 4. $4(6-5)=4\times 1=4$, &c. 5. $14-9=5$, &c. 6. $72+40-105=7$, &c. 7. $280+56+56-140=252$, &c. 8. $16+48+36=100=(4+6)^2=10^2$, &c.; $\frac{1}{4}+\frac{2}{6}+\frac{1}{9}=\frac{25}{36}$. 9. $16-48+36=16+36-48=4=(-4+6)^2=(6-4)^2=2^2$, &c.; $\frac{9}{16}-\frac{12}{12}+\frac{4}{9}=\frac{9}{16}-1+\frac{4}{9}=\frac{9}{16}+\frac{4}{9}-1=\frac{1}{144}=(\frac{3}{4}-\frac{2}{3})^2=(\frac{1}{12})^2$. 10. $4\,.\,3\,.\,2\,.\,1=24$, &c. 11. $s=\frac{1}{2}(4+7+9)=10$, and $10\,.\,6\,.\,3\,.\,1=180$, &c. 12. $64-16=48$, &c. 13. $125-15=110$, &c. 14. $6+6-12+0=0$. 15. $1+16-9-4+5+0=9$. . . . 17. $\sqrt{36}+\sqrt[3]{8}+2+3=13$. . . . 23. $(11\,.\,1+1\,.\,1)^2=(11+1)^2=12^2=144$. . . . 27. $12-4$ and $12\div 4$; $8-3=5$, &c. 28. $3a^2+5a^2=8a^2$. 29. $3a^3-3a^3+4a^3-2a^3=2a^3$. 30. $4x^2-3b^2+5x^2+2b^2-3x^2=6x^2-b^2$. . . . 32. $4-\frac{8}{16}=\dfrac{7}{4-2}$ or $3\frac{1}{2}=3\frac{1}{2}$. . . . 35. $y=\frac{5}{4}-\sqrt{\dfrac{\frac{5}{4}}{\frac{4}{5}}}=\frac{5}{4}-\sqrt{\frac{25}{16}}=\frac{5}{4}-\frac{5}{4}=0$. 36. $y=\left(\dfrac{15}{5}-2\right)(\frac{1}{20})$, or $y=(3-2)\,.\,\frac{1}{20}=\frac{1}{20}$, &c. 37. $y=5\sqrt{81}-\frac{1}{2}\sqrt{64}=41$, &c. 41. $\dfrac{12+12}{6+0}+\dfrac{80}{8}=14$; $\sqrt{(72-8)}+0+\sqrt{(20-4)}+1=8+4+1=13$. 42. $\dfrac{125}{4+1}-\dfrac{8}{1}+\dfrac{16+12}{4+2+1}-10+3=25-8+4-10=14$; $\dfrac{25+16-1}{5+4+1}+\dfrac{40}{8+2}-\dfrac{100-80}{4+1}=4+4-4=4$; $\dfrac{24+160+16}{5+4-2+1}-\dfrac{125-64-8+7}{32}=25-1\frac{7}{8}=23\frac{1}{8}$. . . . 46. $s=21$; $\sqrt{21\,.\,6\,.\,7\,.\,8}=\sqrt{7056}=84$; $s=60$; $\sqrt{(60\,.\,10\,.\,20\,.\,30)}=600$. 47. $1-\{1-(-5)\}+12-(-27)+2-26=1-6+12+27-24=10$. 48. $\frac{1}{6}(4\,.\,5\,.\,6+4\,.\,3\,.\,2)+\frac{2}{3}(3\,.\,4\,.\,5)=24+40=64$. 49. $\sqrt[3]{9\,.\,1\,.\,\sqrt{9}}$

$= \sqrt[3]{27} = 3$; $\sqrt{(16+8+1)} = \sqrt{25} = 5$; $\sqrt{\frac{36-10+6}{6+5-3}} + \sqrt[3]{\frac{18+10-1}{15-9+2}} + \sqrt[4]{\frac{50+27+4}{20-\sqrt[3]{8}-2}} = 2 + \frac{3}{2} + \frac{3}{2} = 5$. In 55, 58, 59 let $a=8$, $b=5$, then $13 \,.\, 13 = 64+80+25 = 169 = 13^2$; $(8+5)+(8-5) = 13+3 = 16 = 2 \,.\, 8$; $(8+5)-(8-5) = 13-3 = 10 = 2 \,.\, 5$. Again if $a=7$, $b=12$, then $19 \,.\, 19 = 49+168+144 = 361 = 19^2$, and $(7+12)+(7-12) = 14 = 2.7$; $(7+12)-(7-12) = 19-(-5) = 19+5$ (Art. 6) $= 24 = 2 \,.\, 12$. In 56 and 60 let $x=9$, $y=6$, then $15 \,.\, 3 = 81-36$ or $45 = 45$; and $(9-4)(9+7) = 81+27-28$, or $5 \,.\, 16 = 81-1 = 80$. In 57 let $n=11$, $m=7$, then $4 \,.\, 4 = 121-154+49 = 121+49-154$; or $16 = 170-154$. 61. It is 5 minutes less than nothing, or -5^m, past 12, that is, it wants 5^m of 12. 62. He is worth £500 less than nothing, or $-$£500, that is, he is £500 in debt after all he has is paid away. 64. It is one degree less than nothing south, that is, it is 1° north of the equator. 63. The gain is £20 less than nothing, or $-$£20, that is, there is a loss of £20.

II. ADDITION, p. 21.

18. $a+b$. 19. $2a$. 20. $x+x^2$. 21. $n+\sqrt{n}$. 22. $-a$. 23. -1. 24. 0. 25. $3d+4c$. 26. $6x-3y$. 27. $\sqrt{x}-2\sqrt[3]{x}$. 28. $-a^2-a$. 29. $a^2x^3+ax^3$. 30. a. 31. $2a-b$. 32. p. 33. $6+4x+x^2$. 34. a^2+x^2. 35. 2. 36. $-11x^3-5x^2$. 37. $2a+5a^2$. 38. a^2+a^3. 39. b^3+3b. 40. $4a^n+\sqrt[n]{x}-\frac{3}{5}y$. 41. $-6a$; $6a$. 42. $5axy+7ax$. 43. -4. 44. $5x(a+b)$. 45. -19. 46. x; a. 47. $(a+x)y$. 48. $5x+b$. 49. $b^4+4-\sqrt{x}$. 50. $6a^m+x^p$. 51. $9a^2-7a+10$. 52. $a+b+c$. 53. $2axy$. 54. $2a^n+3a^{n+1}+1$. 55. $2a+2b+2c+2d$.

III. SUBTRACTION, p. 24.

9. $-13a$. 10. $-5a$. 11. $13a$. 12. $5a$. 13. $2b$. 14. $-2b$. 15. $b-c$. 16. $2a$. 17. 0. 18. x^3-x^2. 19. $ab-a$. 20. $12+a+3b$. 21. $4x-4$. 22. $\sqrt{x}$. 23. a^2-a. 24. b. 25. $3ab.-3a$. 26. $2x-x^2$. 27. -1. 28. $-2a$. 29. $10a$. 30. $2a$. 31. $-10a$. 32. -1. 33. x. 34. $-x$. 35. $-\frac{1}{2}x$. 36. $\frac{1}{2}x$. 37. -3. 38. 0. 39. $2-2a$. 40. $a-b$. 41. 15. 42. $\frac{3}{4}a-\frac{3}{4}b$.

43. -17. 44. $\sqrt[3]{x}-\sqrt[5]{x}$. 45. $4ab$. 46. $\sqrt{x}\,(a-1)-a+1$. 47. a. 48. $-a$. 49. $2+a$. 50. $a-2$. 51. $-2a$. 52. -2. 53. $2a$. 54. 2. 55. b. 56. $x-7-(2x-5+x)=x-7-2x+5-x=-2x-2$. 57. $a-(b-c-x)+(b-x+2b)=a-b+c+x+3b-x=a+2b+c$. 58. $a-\{a-(a-a+x)\}=a-(a-a+a-x)=a-a+a-a+x=x$. 59. $x^2+2xy+y^2-(x^2+xy-y^2-2xy+x^2+y^2)=3xy-x^2+y^2$. 60. $2a\sqrt{(x-y)}-bxy-a\sqrt{(x-y)}+bxy=a\sqrt{(x-y)}$. . . . 66. $1-[1-(1-1+x)]=1-(1-1+1-x)=x$. 67. $6a-\{2a-[5a-3a]\}=6a-(2a-2a)=6a$. 68. $3a-[a+b-\{a+b+c-a-b-c-d\}]=3a-(a+b+d)=2a-b-d$.

IV. MULTIPLICATION, p. 35.

1. ab. 2. npx. 3. a^3. 4. a^5x^3. 5. $56x^2y$. 6. $abcdef$. 7. $-48cd$. 8. $40x^6$. 9. $10a^5$. 10. $-a$. 11. $11a$. 12. $mnpq$. 13. $2ax$. 14. $\frac{1}{2}ab^5$. 15. a^{n+p}. 16. $12b^{2n}$. 17. $8b^7$. 18. b^{n+1}. 19. x^{2n}. 20. a^{2n-1}. 21. x^{n+3}. 22. a^{2n+1}. 23. $a^{n+2}x^{n+3}$. 24. a. 25. 60. 26. $\frac{1}{4}a^2$. 27. $2b$. 28. $20x$. 29. $\frac{1}{9}a^5x^8$. 30. $2a^2$. 31. $8ab\sqrt{ax}$. 32. $a^{15}x^5y^5$. 33. $8x^2-4bx+6d^2x$. 34. $-21a^5+12a^4bx-9a^3d+24a^3$. 35. $a^2b+ab^2x+abdey$. 36. $-28a^4x+16a^4x^2-8a^3bxy$. 37. $12a^2bx-6ab^3y+21a^2bd-15aby$. 38. $8ay\sqrt{x}-6ax^3y^2+10ay\sqrt[3]{x}$. 39. $\frac{2}{3}xy^4-x^2y^2+4ax^2y-xy$. 40. $my^2-3mxy+2amyz$. 41. $-b^3y^3z^2+abdy^3z-2by^5z$. 42. $a^2+2ax+x^2$. 43. $a^2-2ax+x^2$. 44. a^2-x^2. 45. $36-a^2$. 46. $2r^2-10r+12$. 47. n^2-16. 48. $16a^2-4a+\frac{1}{4}$. 49. $m^2+mn+mp+np$. 50. $ac+bc+ad+bd$. 51. $n^2-\frac{1}{16}$. 52. $\frac{1}{16}a^2-\frac{1}{2}a+1$. 53. $9b^2+12by+4y^2$. 54. $25x^4-9y^6$. 55. $a^8+2a^4b+b^2$. 56. $x^2+8x+15$. 57. $x^2-7x-44$. 58. $x^2+7x-44$. 59. 41. 60. $a^{n+1}+ab^n-a^nb-b^{n+1}$. 61. $a^{2n}+2a^nb^n+b^{2n}$. 62. $1-2=-1$. 63. 64, 65, see Alg. p. 327. 66, 67, using det. coef. we get $1\pm0\pm0\pm0+64$; and $1+0+0+0+0-6+5$; that is x^4+64 and $1-6x^5+5x^6$. 68. Arranging by powers of a we have $a^4-2a^3b+4a^2b^2-8ab^3+16b^4$ and $a+2b$; then using the det. coef. all the terms disappear except the first and last which are 1 and 32, that is a^5+32b^5. 69. The method of det. coef. does not apply here; the partial products are $35a^2b^2+21a^2bc-28ab^2c$, $-90a^2bc-54a^2c^2+72abc^2$, $10ab^2c+6abc^2-8b^2c^2$. 70. Here x^1 only has the coef. 0. 71. Here all the powers of x ap-

pear. 72. Squaring by det. coef. we find $1-6+13-12+4$; and using these with $1+6+1$ we find that the second term only is wanting. 73, 74 are very easy by det. coef.; the series of powers is complete. 75. Arranging by the powers of x (Art. 39) and putting zero for the coef. of x^2 which is wanting, the coef. are $3 \pm 0 + 2 - 5$ and $7 - 1 + 2$. 76. The powers are complete. 77. The third, second, and first powers of x vanish. 78. Put $4x^2$ in the second place; all the powers appear in the answer. 79. There is no gap in the series of powers. 80. Series of powers without gap. 81. Here b^1 in the first factor and b^2 in the second are wanting; the det. coef. $\therefore$ are $1-3+0+4$ and $1+0+3-4$; no term vanishes. 82. Putting zeros for the coef. of the wanting terms we have $5+0-3-2$ and $3+0+2-5$; in the result the second term is wanting, that which should contain a^6b^1; the other powers appear. 83. Here $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$, in which $3x^2y+3xy^2=3xy\ (x+y)\ \therefore$ &c. 84. Here $a^3+a^2b+ab^2+b^3=a^3+ab^2+a^2b+b^3=a\ (a^2+b^2)+b\ (a^2+b^2)=(a+b)\ (a^2+b^2). \therefore (a^3+a^2b+ab^2+b^3)\ (a-b)=(a+b)\ (a^2+b^2)\ (a-b)=(a^2+b^2)\ (a+b)\ (a-b)=(a^2+b^2)\ (a^2-b^2)$. 85. Here (note, p. 31 Alg.) $(a^3+3a+1)^2=\{a\ (a+3)+1\}^2=a^2\ (a+3)^2+2a\ (a+3)+1=a\{a\ (a+3)^2+2(a+3)\}+1=a\{\ (a+3)\ .\ a\ .\ (a+3)+2(a+3)\}+1=a\{(a+3)[a(a+3)+2]\}+1=a\{(a+3)\ .\ [a(a+2)+a+2]\}+1=a\{(a+3)[(a+2)\ (a+1)\ ]\}+1=a\ \ (a+3)\ \ (a+2)\ (a+1)+1$ for $3a=2a+a$. Otherwise; both forms produce the same result, namely, $a^4+6a^3+11a^2+6a+1$. 86. $(a+b+c)^3-(a^3+b^3+c^3)=(a+b)^3+c^3+3c(a+b)(a+b+c)-(a^3+b^3+c^3)=a^3+b^3+3ab\ (a+b)+c^3+3c(a+b)\ (a+b+c)-(a^3+b^3+c^3)=3ab\ (a+b)\ +3c\ (a+b)\ (a+b+c)=3\ (a+b)\ [ab+c(a+b+c)]=3(a+b)[(a+c)b+(a+c)c]=3(a+b)\{(a+c)\ (b+c)\}=3(a+b)\ (a+c)\ (b+c)$. 87, 88, 89, 90, present no difficulty.

V. DIVISION, p. 46.

1. b. 2. x^2. 3. 1. 4. $4x^2$. 5. -1. 6. $5a$. 7. $3b$. 8. $3abc$. 9. 1. 10. a. 11. 4. 12. b^n. 13. a^{n-1}. 14. -1. 15. $3b$. 16. $3bde$. 17. x^{a-b}. 18. $8ay$. 19. a. 20. $5a^n$. 21. 1. 22. -1. 23. a^2. 24. $\sqrt{2}$. 25. a^{-2}. 26. $3a^{n-2}x^2y$. 27. $\sqrt{8a}$. 28. $4a$. 29. $-5b^3y$. 30. $\dfrac{3ax}{b}$. 31. $\frac{4}{3}$. 32. $\frac{1}{4}$.

33. $\frac{m}{p}$. 34. x. 35. $x^{-(n+1)}$. 36. $\frac{bc}{d}$. 37. $2\sqrt{x}$. 38. $b^{-3} = \frac{1}{b^3}$. 39. $2\frac{a}{b}$. 40. $-3\frac{a}{b}$. 41. a^{p-q}. 42. $\frac{a^7}{x^4}$. 43. $\frac{3}{2}\cdot\frac{b}{a}$. 44. $\frac{1}{a^2}=a^{-2}$. 45. $\frac{1}{5}b^3q^3y^3$. 46. $4\frac{p}{q}$. 47. $4a^7b^4x^2y$. 48. 1. 49 $-\frac{1}{a}$. 50. $\frac{a}{b}$. 51. $3by+y^2-4$. 52. $c+4x-3a$. 53. $3abx-4y-5a^3b^2x^2+a^2xy^2$. 54. $\frac{1}{2}(a+b)$. 55. $2a-a^2-7+3x+1$. 56. $b+by+by^2+by^3+$ &c. 57. $3a-7x+10y^2-2z+1$. 58. $\frac{7}{4}.nx^2-2\frac{y^3}{n}-\frac{3by^2}{n}+\frac{13}{4}.m$. 59. $ay-2x^2y-4a^5x^{n-3}$. 60. $a-b$. 61. $x+7$. 62. $x+y$. 63. $m-6$. 64. $n+p$. 65. $\sqrt{a}-1$.

66.
$$\begin{array}{ll}
x^3-9x^2+27x-27 & (x-3, \text{ divisor.}\\
x^3-3x^2 & \overline{x^2-6x+9}, \text{ quotient.}\\
\hline
\quad -6x^2+27x & \\
\quad -6x^2+18x & \\
\hline
\qquad\quad 9x-27 & \\
\qquad\quad 9x-27 & \\
\hline
\qquad\qquad 0 &
\end{array}$$

68.
$$\begin{array}{ll}
a+x)\,a^{n+1}+2a^nx+2a^{n-1}x^2+2a^{n-2}x^3 \text{ \&c.} & (a^n+a^{n-1}x\\
\quad\;\; a^{n+1}+a^nx & +a^{n-2}x^2+\\
\hline
\qquad\qquad a^nx+2a^{n-1}x^2 & \text{\&c.}\\
\qquad\qquad a^nx+\ a^{n-1}x^2 & \\
\hline
\qquad\qquad\qquad a^{n-1}x^2+2a^{n-2}x^3+\text{\&c.} & \\
\qquad\qquad\qquad a^{n-1}x^2+a^{n-2}x^3 & \\
\hline
\qquad\qquad\qquad\qquad a^{n-2}x^3 \text{ \&c.} &
\end{array}$$

69.
$$\begin{array}{ll}
1-5x+10x^2-10x^3+5x^4-x^5 & (1-2x+x^2, \text{ divisor.}\\
1-2x+x^2 & \overline{1-3x+3x^2-x^3},\\
\hline
\quad -3x+9x^2-10x^3 & \qquad \text{quotient.}\\
\quad -3x+6x^2-3x^3 & \\
\hline
\qquad\quad 3x^2-7x^3+5x^4 & \\
\qquad\quad 3x^2-6x^3+3x^4 & \\
\hline
\qquad\qquad -x^3+2x^4-x^5 & \\
\qquad\qquad -x^3+2x^4-x^5 & \\
\hline
\qquad\qquad\qquad 0 &
\end{array}$$

70. $4 + 4 - 29 - 21 \; (2 - 3$

$-4 - 6 \qquad \overline{-2 - 5 - 7}$

$10 - 29 \qquad$ i.e.

$-10 - 15 \qquad 2x^2 + 5x - 7$

$-14 - 21$

$-14 - 21$

0

72. $x^4 - a^4 \; (\, x - a$

$x^4 - ax^3 \qquad \overline{x^3 + ax^2 + a^2x + a^3}$

$ax^3 - a^4$

$ax^3 - a^2x^2$

$a^2x^2 - a^4$

$a^2x^2 - a^3x$

$a^3x - a^4$

$a^3x - a^4$

74. Note that $x^{pq-q} = x^{q(p-1)}$, $x^{pq-q-q} = x^{q(p-2)}$; the successive factors $p-2$, $p-3$, $p-4$, &c., diminish continually and the indices of x become at last $5q$, $4q$, $3q$, $2q$, $1q$, and $0.q$, so that $x^0 = 1$.

84. $8 + 6 - 15 \pm 0 + 1 \; (4 - 3 - 1 \qquad$ 1st rem. 12

$-8 + 6 + 2 \qquad \overline{-2 - 3 + 1} \qquad$ 2nd „ -4.

$-12 + 9 + 3 \qquad$ i.e.

$4 - 3 - 1 \qquad 2 + 3 - 1$

$0 + 0 + 0 + 0 + 0$

85. $6 - 1 - 13 + 10 - 2 \; (2 - 3 + 1 \qquad$ Rems. $+8$ and -4.

$-6 + 9 - 3 \qquad \overline{-3 - 4 + 2}$

$-8 + 12 - 4 \qquad$ i.e.

$4 - 6 + 2 \qquad 3 + 4 - 2$

86. $10 - 48 + 51 + 4 - 15 \; (-5 + 4 + 3$, arranging by indices.

$-10 + 8 + 6 \qquad \overline{2 - 8 + 5}$

$40 - 32 - 24 \qquad$ i.e.

$-25 + 20 + 15 \qquad -2 + 8 - 5$

1st rem. -40; 2nd, $+25$.

83.
$$\begin{array}{l}
12a^2 + 26ab - 36ac + 18ad - 10b^2 + 29bc - 6bd - 21c^2 + 9cd \ (6a - 2b + 3c \\
\underline{12a^2 - \ \ 4ab + \ \ 6ac} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \overline{2a + 5b - 7c + 3d} \\
\qquad 30ab - 42ac + 18ad - 10b^2 + 29bc \\
\qquad \underline{30ab - 10b^2 + 15bc} \\
\qquad\qquad - 42ac + 14bc + 18ad - \ \ 6bd - 21c^2 \\
\qquad\qquad \underline{- 42ac + 14bc - 21c^2} \\
\qquad\qquad\qquad\qquad 18ad - \ \ 6bd + \ \ 9cd \\
\qquad\qquad\qquad\qquad \underline{18ad - \ \ 6bd + \ \ 9cd} \\
\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

87.
$$\begin{array}{l}
- 40y^5 + 68xy^4 + 25x^2y^3 + 21x^3y^2 - 18x^4y - 56x^5 \ (5y^2 - 6xy - 8x^2 \\
\ \ 40y^5 - 48xy^4 - 64x^2y^3 \qquad\qquad\qquad\qquad\qquad\qquad \overline{8y^3 - 4xy^2 + 3x^2y - 7x^3} \\
\qquad - 20xy^4 + 24x^2y^3 + 32x^3y^2 \qquad\qquad\qquad\qquad\qquad i.e., \\
\qquad\qquad\qquad 15x^2y^3 - 18x^3y^2 - 24x^4y \qquad - 8y^3 + 4xy^2 - 3x^2y + 7x^3 \\
\qquad\qquad\qquad\qquad \underline{- 35x^3y^2 + 42x^4y + 56x^5}
\end{array}$$

1st remainder $20xy^4$; 2nd, $-15x^2y^3$; 3nd, $35x^3y^2$.

88. The work is as in Ex. 69.

89. Arrange the terms by the powers of a and take the co-efficients.

$$\begin{array}{rl}
12+8-13-9+2-1 & (3+2-1 \\
-12-8+4 \qquad\qquad & \overline{-4+3+1} \\
9+6-3 \qquad & \quad i.e. \\
3+2-1 & \quad 4-3-1 \\
\hline
1-2 &
\end{array}$$

90. Arranging according to the indices we have

$$\begin{array}{ll}
1\pm0-4+5\pm0-10+19-20+14-5 & (1+2-3-4+5 \\
 & \overline{-1+2-3+3-2+1} \\
-1-2+3+4-5 & \text{or } 1-2+3-3+2-1 \\
\quad 2+4-6-8+10 & \\
\qquad -3-6+9+12-15 & \text{Rems. } -2,+3,-3,+2,-1. \\
\qquad\quad +3+6-\ 9-12+15 & \\
\qquad\qquad -2-\ 4+\ 6+\ 8-10 & \\
\qquad\qquad\quad +\ 1+\ 2-\ 3-\ 4+5 & \\
\hline
\end{array}$$

Miscellaneous Exercises, p. 49.

1. Twice the greater; for $(a+b)+(a-b)=2a$. 2. Twice the less; for $(a+b)-(a-b)=2b$ (see Ex. 58 and 59, p. 17). 3. $x+12$. 4. $y-d$. 5. Twice the product for $(a+x)^2-(a^2+x^2)=2\,ax$ (Euc. II. 4). 6. Four times the product; for $(a+x)^2-(a-x)^2=4ax$ (Euc. II. 8). 7. Twice the sum of the squares; for $(a+x)^2+(a-x)^2=2\,(a^2+x^2)$. 8. $pq+r$. 9. $2pq$. 10. 36. 11. $\frac{£a}{x}$. 12. $10x+y$; $100x+10y+z$. 13. $a+x$ and $a-x$; $8+x$ and $8-x$; $2n+8$ and $2n-8$; $\surd x+10$ and $\surd x-10$; $9a^2+10x$ and $9a^2-10x$ (note p. 31; Euc. II. 5). 14. $(x-y)\,(x^2+xy+y^2)$; $(x+y)\,(x-y)\,(x^2+y^2)$; $(a-x)\,(a^2+ax+x^2)\,(a+x)\,(a^2-ax+x^2)$, because $a^3+x^3=(a+x)\,(a^2-ax+x^2)$; $(a^4+x^4)\,(a^2+x^2)\,(a+x)\,(a-x)$; $(a-x)\,(a+x)\,(a^2+x^2)\,(a^4+x^4)\,(a^8+x^8)$. 15. $\frac{a}{x}$. 16. $36a^2-48ax+16x^2$; $4b^2-20by^3+25y^6$; $\frac{1}{4}a^2-8a+64$. 17. x^2. 18. $2\,(s-a)=b+c-a$; $2\,(s-b)=a+c-b$; $2\,(s-c)=a+b-c$. 19. Let a be the greater and b the less; then $\frac{1}{2}\,(a+b)-b=\frac{1}{2}\,(a-b)$; also $a-\frac{1}{2}\,(a+b)=\frac{1}{2}\,(a-b)$ ∴ &c. 20. Call the work 1; then the parts are $\frac{1}{12}, \frac{5}{12}, \frac{7}{12}$ and $\frac{x}{12}$. 21. $a+b-c$ and $a-b+c$; $b+c+a$ and $b+c-a$. 22. First, the factors are (note p. 32 Alg.) $2ab+a^2+b^2-c^2$, and $2\,ab-$

$a^2 - b^2 + c^2$; again these are equivalent to $(a+b)^2 - c^2$ and $c^2 - (a-b)^2$, so that the expression is the same as $(a+b+c)(b+c-a)(a-b+c)(a+b-c)$. 23. 245 and 111 (note, p. 25). 24. 342 and 94. 25. $42x$ and $38x$ are the spaces passed over; then $42x + 38x = 400$ miles. 26. If A who gained had had 312 votes less, or B who lost 312 more, there would have been an equal number for each; hence the principle of the note p. 25 applies. See Ex. 1 and 2 of this section. The numbers are 1870 and 1558. 27. Half the sum of the quantities. For calling s half the sum of a, b, c, or putting $2s = a+b+c$ we have $s - a = \frac{1}{2}(b+c-a)$; $s - b = \frac{1}{2}(a+c-b)$; $s - c = \frac{1}{2}(a+b-c)$; the sum of these three remainders is $\frac{1}{2}(a+b+c)$. Or, $3s - a - b - c = 3s - (a+b+c) = 3s - 2s = s = \frac{1}{2}(a+b+c)$. 28. Here $s = \frac{1}{2}(a+b+c+d)$, or, $2s = a+b+c+d$; then $s - a = -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$; $s - b = \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$ &c., and adding the remainders the sum is $a+b+c+d$, or $2s = 4s - (a+b+c+d) = 4s - 2s$. Again, in the case of five quantities, a, b, c, d, e, we shall find similarly that the sum is $5s - 2s = 3s = \frac{3}{2}(a+b+c+d+e)$. In the case of six quantities the sum would be $4s = 2(a+b+\text{\&c.})$ It will be a useful exercise to illustrate these and other cases with particular numbers. 29. Here x^3 becomes $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. 30. Here $(a+x)(a-x)$ becomes $(2a+b)(-b) = -2ab - b^2 = -(2ab+b)$. 31. $ax - bx = (a-b)x = (a-b)(a-b) = a^2 - 2ab + b^2$. 32. Developing as in Ex. 29 and subtracting, the first terms are found to be $4x^3h$, $5x^4h$, $6x^5h$.

VI. MEASURES AND MULTIPLES, G. C. M., p. 63.

1, 2, 3. From the first remainders we get respectively a^2, $2x^2$ and $2x$. 4. Suppress a, a factor of the first but not of the second, and $2b^2$ a factor of the second but not of the first; double the terms of the first quantity to render the first term divisible by $6a^3$; we then have $30a^4 + 20a^3b + 8a^2b^2 + 12ab^3 - 6b^4$ and $6a^3 + 19a^2b + 8ab^2 - 5b^3$. Then detaching the coeffs. the work is as under :

$$
\begin{array}{rrrrrl}
6+19+8-5)30 & +20 & +\ \ 8 & +\ \ 12 & -6(& -5+25 \\
-30 & -75 & -\ \ 32 & +\ \ 37 & -6 & \\
 & 2 & & & & \\
\hline
 & -150 & -\ \ 64 & +\ \ 74 & -12 & \\
 & & +411 & +274 & -137 &
\end{array}
$$

$$\text{or, } 137b^2\,(3a^2+2ab-b^2)$$

$$\begin{array}{l} 3+2-1)6+19+\ \ 8-5(-2-5 \\ \qquad\quad -6-\ \ 4+\ \ 2 \\ \qquad\qquad\ \ -15-10+5 \end{array}$$

Here dividing and setting down 5 with sign changed, and adding (subtracting) mentally we have sum (remainder) $0-75-32+37$ and -6 brought down, doubling this to make first term divisible we have 6 in 150 contained 25 times; then multiplying the terms of the divisor by 25 and subtracting we have for remainders, $+411$, &c.; rejecting $137b^2$ we divide again and have quotient, $-2-5$; that is, $2a+5b$; so that $3+2-1$ *i.e.* $3a^2+2ab-b^2$ is the G. C. M. required. 5. Arrange by the powers of x, double the dividend; the quantities then are

$$\begin{array}{l} 12x^5-\ \ 8x^4-22x^3-\ \ 6x^2-\ \ 6x-2\ \ (4x^4+2x^3-18x^2+3x-5 \\ -12x^5-\ \ 6x^4+54x^3-\ \ 9x^2\ +15x \qquad\qquad -3x+7 \\ \hline \qquad -14x^4+32x^3-\ \ 15x^2\ +\ \ 9x-2 \\ \qquad\quad 2 \\ \hline \qquad -28x^4+64x^3-\ \ 30x^2\ +18x-4 \\ \qquad +28x^4+14x^3-126x^2\ +21x-35 \\ \hline \qquad\qquad\qquad 78x^3-156x^2\ +39x-39=39(2x^3-4x^2+x-1) \end{array}$$

Reject 39

$$\begin{array}{l} 4x^4+\ \ \ 2x^3-18x^2+3x-5(2x^3-4x^2+x-1 \\ -4x^4+\ \ \ 8x^3-\ \ 2x^2+2x \qquad\quad -2x\ -5\ ;\ i.e. \\ \hline \qquad\quad 10x^3-20x^2+5x-5 \qquad\quad 2x\ +5 \\ \qquad -10x^3+20x^2-5x+5 \\ \hline \qquad\qquad\qquad 0 \end{array}$$

6. The expressions are the same as $(y+4)\,(y+1)$; $(y+4)$ $(y-2)$, and $(y+4)\,(y+3)$ in which $y+4$ is the G. C. M.; otherwise divide the first by the second and the third by the result.

7. Arranging we have $14x^3+ax^2-a^2x+a^3$ and $7x^3+4ax^2-2a^2x+a^3$. Reject a from first remainder and the new divisor becomes $7x^2-3ax+a^2$; and the quotient is $x+a$ without remainder.

8. Multiply the dividend by 3, the first remainder by 3, and when the second step in the division is completed the quotient (sign changed, Alg. p. 41) is $-4a^2+28$, and second remainder or new divisor, $-135a^3+1106a^2-723a+820$. The

first divisor is then multiplied by 9; this gets for new dividend $135a^4-81a^3+423a^2-189a+252$; dividing this by the new divisor the first term of the quotient (sign changed) is a; and first remainder $1025a^3-300a^2+631a+252$. This must be now multiplied by 27 to avoid fractions; we thus get $27675a^3-8100a^2+17037a+6804$. The next term of the quotient (sign changed) is now 205, and second remainder $218630a^2-131178a+174904=43726\,(5a^2-3a+4)$; 43726 is suppressed as no factor of it is a factor of the divisor. We have then for new divisor $5a^2-3a+4$, and for dividend $-135a^3+1106a^2-723a+820$. The quotient (sign changed) is $27a-41$ without remainder.

9. Multiply the dividend by 7; x is the first quotient; the divisor is then multiplied by 11 for a new dividend, and the factor 114 suppressed, giving x^2-2x+3 for divisor.

12. Making the first the divisor we get quotient 1 and remainder $(p-q)\,x^3+(p-q)\,x^2-(p-q)\,x-(p-q)$; omitting $p-q$ we get $x^3+x^2-x-1=(x^2-1)\,(x+1)=(x+1)\,(x-1)\,(x+1)$, and then rejecting $x+1$, not a factor of the dividend, the greatest common divisor of this and the preceding divisor is the highest measure required; that is x^2-1 is the G. C. M. Otherwise; we have, by resolution into factors, $(x^2-1)\times(x^2-px+q)$ and $(x^2-1)\,(x^2-qx+p)$; but x^2-px+q and x^2-qx+p have no common divisor, so that x^2-1 is the divisor required.

13. From the first expression omit $4x^2$ and from the second $2x$; then multiply the first by 3 and divide the product by the second; the remainder $-5ax^2+14a^2x-9a^3$ being multiplied by 3, gives $-5a$ for second term of quotient with remainder $-8a^2x+8a^3$; from this omit $-8a^2$ and the divisor is $x-a$, and dividend $3x^2-10ax+7a^2$, which gives the quotient $3x-7a$ without remainder.

14. First quotient $4x+3$; second remainder $x-5$; second quotient $x-2$; $\therefore$ G. C. M. is $x-5$.

10. We shall shew the work of this Ex. by the ordinary uncontracted process:

$$9a^6+11a^4+18a^3+42a^2-8)\,18a^6-5a^4-18a^3+33a^2-4\,(2, \text{ quotient.}$$
$$18a^6+22a^4+36a^3+84a^2-16$$
$$\div(-3) \qquad -3)\,\overline{-27a^4-54a^3-51a^2+12}$$
$$\text{Divisor,} \quad 9a^4+18a^3+17a^2-4)\,9a^6+11a^4+18a^3+42a^2-8\,(a^2-2a+10, \text{ quotient.}$$
$$9a^6+18a^5+17a^4-4a^2$$
$$-18a^5-6a^4+18a^3+46a^2-8$$
$$-18a^5-36a^4-34a^3+8a$$
$$\times 3 \qquad 30a^4+52a^3+46a^2-8a-8$$
$$3$$
$$\overline{90a^4+156a^3+138a^2-24a-24}$$
$$90a^4+180a^3+170a^2-40$$
$$\div(-8) \qquad -8)\,\overline{-24a^3-32a^2-24a+16}$$
$$\text{Divisor,} \quad 3a^3+4a^2+3a-2)\,9a^4+18a^3+17a^2-4\,(3a+2, \text{ quotient.}$$
$$9a^4+12a^3+9a^2-6a$$
$$6a^3+8a^2+6a-4$$
$$6a^3+8a^2+6a-4$$
$$0$$

11. Arrange by the powers of x, divide the divisor by 2, multiply the dividend by 3, and detach the coeffs., placing the divisor and quotient on the right in the first operation; the work will be as under:

```
              3 + 12 -  9 - 48 +  33 + 36 - 27 (3 + 10 - 6 - 24 + 11 + 6, divisor.
            - 3 - 10 +  6 + 24 -  11 -  6      -1 - 2
                   2 -  3 - 24 +  22 + 30 - 27
x 3                3
                   6 -  9 - 72 +  66 + 90 - 81            x divisor by 29.
                 - 6 - 20 + 12 +  48 - 22 - 12
                   - 29 - 60 + 114 + 68 - 93)  87 + 290 - 174 - 696 + 319 + 174 (3 + 55, quotient.
                                             - 87 - 180 + 342 + 204 - 279
÷ 2                                             2) 110 + 168 - 492 +  40 + 174
                                                     55 +  84 - 246 +  20 +  87
x 29                                                 29
                                                   1595 + 2436 - 7134 +  580 + 2523
                                                 - 1595 - 3300 + 6270 + 3740 - 5115
÷ (- 864)                                       -864)  -  864 -  864 + 4320 - 2592
                  - 29 - 60 + 114 + 68 - 93          (+ 1 + 1 - 5 + 3, divisor.
                  + 29 + 29 - 145 + 87                  29 + 31          quotient.
                       + 31 + 31 - 155 + 93   ∴ G. C. M. is x³ + x² - 5x + 3.
                              0
```

$\therefore$ G. C. M. is $x^3 + x^2 - 5x + 3$.

15. Here supplying the place of the wanting powers the work will be as under:

```
        16 ± 0 − 53 + 45 + 6  (8 − 30 + 31 ± 0 − 12
      − 16 + 60 − 62 ± 0 + 24    − 2
÷ 5       5)60 − 115 + 45 + 30          × divisor by 3
            12 − 23 + 9 + 6)  24 − 90 + 93 ± 0 − 36( − 2 + 22, quotient.
                            − 24 + 46 − 18 − 12
                                 − 44 + 75 − 12 − 36
× 6                                  6
                                 − 264 + 450 − 72 − 216
                                 + 264   506 + 198 + 132
÷ (− 14)                         − 14) − 56 + 126 − 84
                  Divisor,          4 − 9 + 6)12 − 23 + 9 + 6( − 3 − 1, quotient.
           Ans. 4x² − 9x + 6                 − 12 + 27 − 18
                                                  − 4 + 9 − 6
```

16. $\qquad 4x^5 + 11x^4 + 81(2x^5 - 11x^2 - 9$, divisor.

$\qquad -4x^5 + 22x^2 + 18 \qquad -2$

$\div 11 \quad 11)\ 11x^4 + 22x^2 + 99$

Divisor, $\quad x^4 + 2x^2 + 9)2x^5 - 11x^2 - 9(-2x$

$\times 4 \qquad 4 \qquad -2x^5 - 4x^3 - 18x$

$\qquad 4x^4 + 8x^2 + 36 \qquad (-4x^3 - 11x^2 - 18x - 9$, divisor.

$\qquad -4x^4 - 11x^3 - 18x^2 - 9x \qquad x$, quotient.

$\qquad -11x^3 - 10x^2 - 9x + 36$

$\times$ divisor by 11

Divisor, $\quad -11x^3 - 10x^2 - 9x + 36) - 44x^3 - 121x^2 - 198x - 99(-4$

$\qquad 44x^3 + 40x^2 + 36x - 144$

$\div(-81) \qquad -81)\ -81x^2 - 162x - 243$

$\qquad -11x^3 - 10x^2 - 9x + 36 \quad (x^2 + 2x + 3$, divisor. Ans.

$\qquad 11x^3 + 22x^2 + 33x \qquad 11x - 12$

$\qquad -12x^2 - 24x - 36$

. . .

17. Arranging the terms in reference to the letter a, the first becomes $4ad\,(q+6) - 7xy\,(q+6)$ or $(4ad - 7xy)(q+6)$. The second equals $3bc\,(q+6) + 5mp\,(q+6)$, or $(3bc + 5mp)(q+6)$. Hence $q+6$ measures both, and the other terms have no common factor.

18. Resolved, the expressions become $b^2(a^2 - x^2) - x^2(a^2 - x^2) = (b^2 - x^2)(a^2 - x^2) = (b^2 - x^2)(a+x)(a-x)$ and $4ab(a-x) - 2x^2(a-x) = (4ab - 2x^2)(a-x)$. Hence $a - x$ measures both, the other terms being prime to one another.

19. $\frac{a^2}{3} + \frac{11a}{6}\sqrt{(a+1)} - a - 1)a^2 - \frac{a}{4} - \frac{1}{4}(3$

$$\frac{a^2 + \frac{11a}{2}\sqrt{(a+1)} - 3a - 3}{-\frac{11a}{2}\sqrt{(a+1)} + \frac{11a}{4} + \frac{11}{4}}$$

or $-\frac{11}{2}\sqrt{(a+1)}\{a-\frac{1}{2}\sqrt{(a+1)}\}$.

$\times$ divisor by 6; $2a^2+11a\sqrt{(a+1)}-6a-6$; omit the factor $\frac{11}{2}\times\sqrt{(a+1)}$

$$\begin{array}{l|l|l} \text{Divisor}, a-\frac{1}{2}\sqrt{(a+1)} & 2a^2+11a\sqrt{(a+1)}-6a-6 & 2a+12\times\sqrt{(a+1)} \\ & 2a^2-\quad a\sqrt{(a+1)} & \\ \hline & 12a\sqrt{(a+1)}-6a-6 & \\ & 12a\sqrt{(a+1)}-6a-6 & \\ \hline & \cdot\qquad\cdot\qquad\cdot & \end{array}$$

$$\left.\begin{array}{l}\sqrt{(a+1)}\,.\,\sqrt{(a+1)}=a+1\\ (-12)\,.\,\frac{1}{2}=-6;(a+1)(-6)\end{array}\right\}$$

20. Resolving, $(6a-21)x^2-(6a-21)b^2-9(6a-21)=(6a-21)(x^2-b^2-9)$; $(6a-21)(x^5-10a)$; $(6a-21)(b^3-2)$; $\therefore$ &c. the second factors having no C. M.

L. C. M., p. 65.

1. Detach $4x^2$. 2. The last quantity is divisible by each of the others (note, p. 48, Alg.) 3. Omit the factors 3, 5, a, b, x; then $7\,.\,1\,.\,9\,.\,2\,.\,x\,.\,b\,.\,a\,.\,3\,.\,5=1890abx$. 4. First find G. C. M.; first quantity $\times$ 4 $\div$ second quantity gives x; remainder $\times(-4)$ gives $40x^2-92x+48$; the second term of quotient is 5 and remainder $-42x+63$ or $-21(2x-3)$; making this divisor, the quotient is $4x+1$ without remainder, so that $2x-3$ is the G. C. M. The second quantity $\div$ this G. C. M. $=4x+1$ $\therefore$ first quantity $\times(4x+1)=$ L. C. M. (Art. 61.) 5. Resolving (Ex. 13, 14, p. 49) we get $(x-a)(x^2+xa+a^2)$ and $(x-a)(x+a)$; $\therefore x-a$ is G. C. M. $\therefore$ first quantity $\div(x-a)=x^2+xa+a^2$; $\therefore$ (Art. 61) $(x^2+xa+a^2)\,.\,(x^2-a^2)=$ L. C. M. 6. The G. C. M. is easily found to be x^2+7; the first quantity $\div x^2+7=x-3$ $\therefore$ &c. (Art. 61.) 7. Dividing and omitting $2x^2$ from first remainder, the G. C. M. is a^2-x^2 $\therefore$ &c. 8. Take the factors; $(x+a)(x^4+a^2x^2+a^4)$ and $(x-a)(x^4+a^2x^2+a^4)$; $\therefore x^4+a^2x^2+a^4$ is G. C. M. $\therefore (x+a)(x-a)(x^4+x^2a^2+a^4)$ is L. C. M.

9. Resolving, $3x^2-11x+6=3x(x-3)-2(x-3)=(3x-2)\times(x-3)$ .... (1)

$2x^2-7x+3=2x(x-3)-(x-3)=(2x-1)\times(x-3)$ .... (2)

Resolving, $6x^2 - 7x + 2 = 3x(2x-1) - 2(2x-1) = (3x-2) \times (2x-1)$ ......(3)

Now, the G. C. M. of (1) and (2) is $x-3$, $\therefore$ L. C. M. is $(3x-2)(2x-1)(x-3)$ ......(4)

Again, of (3) and (4) the G. C. M. is $(3x-2)(2x-1)$, $\therefore$ the L. C. M. required is $(3x-2)(2x-1)(x-3)$.

10. Here $x^3 - 3x^2 + 3x - 1 = (x-1)^3$ ......(1)

$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1) = (x^2-1) \times (x-1) = (x+1)(x-1)(x-1) = (x+1)(x-1)^2$...(2)

$x^4 - 2x^3 + 2x - 1 = x^2(x^2 - 2x + 1) - (x^2 - 2x + 1) = (x^2-1)(x-1)^2 = (x+1)(x-1)(x-1)^2 = (x+1)(x-1)^3$ ......(3)

$x^4 - 2x^3 + 2x^2 - 2x + 1 = x^2(x^2 - 2x + 1) + (x^2 - 2x + 1) = (x^2+1)(x^2 - 2x + 1) = (x^2+1)(x-1)^2$ ....(4)

Now, $(x-1)^3$ is G. C. M. of (1) and (3), and $\therefore$ the L. C. M. is $(x-1)^3(x+1)$ ......(5)

The G. C. M. of (2) and (4) is $(x-1)^2$, $\therefore$ the L. C. M. is $(x-1)^2(x+1)(x^2+1)$ ......(6)

And since the G. C. M. of (5) and (6) is $(x-1)^2(x+1)$, $\therefore$ the L. C. M. required is $(x-1)^3(x+1)(x^2+1)$.

VII. FRACTIONS, p. 79.

1. 1. 2. $3a$. 3. $2acx$. 4. $2a$. 5. $3abx$. 6. $5bx$. 7. $\frac{1}{xy}$. 8. $\frac{1}{2a}$. 9. $\frac{n^2}{2}$. 10. $\frac{1}{4} \cdot \frac{a}{b}$. 11. $\frac{7}{9}$. 12. $\frac{a}{bc}$. 13. $n-1$. 14. $\frac{2ay}{3nx}$. 15. $\frac{a}{a+x}$. 16. $\frac{1}{5cd}$. 17. $\frac{m-q}{(m+q)^2}$. 18. 1. 19. $\frac{1}{(b-x)^2}$. 20. $\frac{1}{4}$. 21. $\frac{3}{4} \cdot \frac{x}{ay}$. 22. $\frac{2}{5} \cdot \frac{ax}{b}$. 23. $\frac{1}{3} \cdot \frac{y}{ab}$. 24. $\frac{1}{a}$. 25. $\frac{a+x}{3b-c}$. 26. $\frac{4bc}{5ax}$. 27. $\frac{n+3}{n-3}$. 28. $\frac{a^2+ab+b^2}{a+b}$. 29. $\frac{a^2+ab+b^2}{a-b}$. 30. $\frac{n^2}{n-2}$. 31. $\frac{7a}{5c}$. 32. $\frac{x-1}{x+2}$. 33 to 42, the G. C. M.s are ax; $5x-2$; $2a^2-3x^2$; $x-a$; $x+7$; $a-b$; $x-1$; $x-3$; $3ab^2x$; $5a^nb^2c^2$. 43. The G. C. M. is found by the following process:

$$3 \pm 0 - 10 \pm 0 + 15 + 8 \,\Big|\, 1 - 2 - 6 + 4 + 13 + 6, \text{ divisor.}$$
$$-3 + 6 + 18 - 12 - 39 - 18 \,\Big|\, -3$$
$$(\div 2) \quad 6 + 8 - 12 - 24 - 10 \qquad \times 3$$
$$\text{Divisor, } 3 + 4 - 6 - 12 - 5)3 - 6 - 18 + 12 + 39 + 18(-1$$
$$-3 - 4 + 6 + 12 + 5$$
$$(\div 2) \quad -10 - 12 + 24 + 44 + 18$$
$$\text{or,} \quad -5 - 6 + 12 + 22 + 9$$
$$(\times 3) \quad 3$$
$$-15 - 18 + 36 + 66 + 27 \,(5$$
$$+15 + 20 - 30 - 60 - 25$$
$$2 + 6 + 6 + 2, \text{ or}$$
$$3 + 4 - 6 - 12 - 5(1 + 3 + 3 + 1, \text{ divisor.}$$
$$-3 - 9 - 9 - 3 \qquad -3 + 5, \text{ or } 3x - 5$$
$$+5 + 15 + 15 + 5$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Hence $x^3 + 3x^2 + 3x + 1$ is the G. C. M.

44. $\frac{15bx}{5b}$. 45. $\frac{a^2 - x^2}{a + x}$. 46. $\frac{60a^2xy^2}{5y^2}$. 47. $\frac{2a^2 - 3a + 1}{a - 1}$.

48. $\frac{a\sqrt{x}}{a}$. 49. $\dfrac{\frac{b(a-x)^3}{(a-x)^2}}{\frac{b}{(a-x)^2}}$. 50. $b(a + x)$. 51. $4c + 3 + \frac{5}{2c}$. 52. $2bx - 3a + \frac{7}{b}$. 53. See Ex. 79, Division. . . .

67. $\frac{mq \pm np}{nq}$. 68. $\frac{m^2 \pm n^2}{mn}$. 69. $2\frac{a^2 + x^2}{a^2 - x^2}$, or $\frac{4ax}{a^2 - x^2}$. 70. $\frac{2}{1 - x^2}$ or $-\frac{2x}{1 - x^2}$. 71. $\frac{5x - 3ay}{7a}$. 72. $\frac{x}{14}$. 73. $\frac{31x}{20}$, or $\frac{x}{20}$. . . .

82. Taking the upper signs we have $\frac{1}{3(1 - x)} + \frac{2 + x}{3(1 + x + x^2)} = \frac{1}{1 - x^3}$; taking the lower we find $\frac{1}{3(1 + x)} + \frac{2 - x}{3(1 - x + x^2)} = \frac{1}{1 + x^3}$. 84. Multiply each numerator by all the denominators except its own, and arrange the terms of the numerator by the indices and traits. 85. Proceeding as in the last Ex. we have for num. $2x^2 + 10x + 12 - 16x^2 - 64x - 48 + 18x^2 + 54x + 36 = 4x^2$; and for denominator $4(x^3 + 6x^2 + 11x + 6) = 4(x + 1)(x + 2)(x + 3)$. 86. The work is like that of 84 and

85; num. $2x^4+7x^3-2x^2-28x-24-(3x^4-8x^3-8x^2+32x-16)+x^4-16x^3-4x^2+64x=-x^3+2x^2+4x-8=-(x^3-2x^2-4x+8)=(-x^2+4)(x-2)=-(x^2-4)(x-2)$; and the denomr. $(4-x^2)(x^2-4)=(-x^2+4)(x^2-4)=-(x^2-4)\times(x^2-4)=-(x^2-4)(x-2)(x+2)$; $\therefore\ \dfrac{-(x^2-4)(x-2)}{-(x^2-4)(x-2)(x+2)}=\dfrac{1}{x+2}$. 89. The com. denr. is plainly $(a-b)(b-c)(c-a)$; working as in 84, 85, 86, and omitting the common factors the numerators come out $a^2-b^2, b^2-c^2, c^2-a^2$, whose sum is zero. 90 to 99 see Art. 75. 100. $\dfrac{a^2-b^2}{6ax}$. 101. $\dfrac{a^3}{27}$. 102. $\frac{1}{2}ab$. 103. $\dfrac{bx}{cy}$. 104. 1. . . . 114. Here $\dfrac{(x-4)(x-5)}{x(x-6)}\times\dfrac{(x-6)(x-7)}{x(x-5)}=\dfrac{(x-4)(x-7)}{x^2}$. . . . 125. a^2. 126. x. 127. $(a-x)^2$. 128. $\dfrac{1}{a^2}$. 129. $\dfrac{(a-b)^2}{c^2}$. 130. $x^8-2+\dfrac{1}{x^8}$. 131. Reduce each to a fractional form and then divide. 132. $\dfrac{(x+1)(x+2)}{(x+1)(x+1)}\times\dfrac{(x+1)(x+4)}{(x+3)(x+4)}=\dfrac{x+2}{x+1}\times\dfrac{x+1}{x+3}=\dfrac{x+2}{x+3}$. 133. Divide, without altering the form of the quantities. 134. Multiply every term by b. . . . 137. Multiply both terms by $1+x$. 138. By simplifying the denomr.; $1\div\dfrac{1}{x^2-7x+12}$. 139. Multiply both terms by a^2. 140. Multiply both terms by 18; this gives $\dfrac{18x^3-11a^2x-2a^3}{18x^2-6ax-12a^2}$. To find the G. C. M. of the terms, divide in the usual way; multiply the first remr. by 3, and from the second reject the factor $3a^2$, then $3x+2a$ is the G. C. M. 141. Reduction of denomr. gets $\dfrac{1}{x+\frac{1}{x}}=\dfrac{1}{\frac{x^2+1}{x}}=\dfrac{x}{x^2+1}$; then $\dfrac{1}{1+\frac{x}{x^2+1}}=\dfrac{1}{\frac{x^2+x+1}{x^2+1}}=\dfrac{x^2+1}{x^2+x+1}$. 142. Reduction of denr. gives $1+\dfrac{x}{x+1}=\dfrac{2x+1}{x+1}$, $\therefore$ &c. 143. Here

$1 + \frac{a}{4-a} = \frac{4}{4-a}$ and the fraction becomes $\frac{1}{a - 1 + \frac{4-a}{4}}$, &c.

144. Here $1 + x + \frac{x}{1+x+x^2} = \frac{1+3x+2x^2+x^3}{1+x+x^2}$, and the denr. becomes $1 + \frac{x+x^2+x^3}{1+3x+2x^2+x^3} = \frac{1+4x+3x^2+2x^3}{1+3x+2x^2+x^3}$; divide x by this. 145. Multiply both terms by $1+ba$, and simplify. 146. Multiply every term by $1-x^2$ the L. C. M. of the denoms. 147. Multiply every term by a^2-b^2. 148. Multiply the terms by a^2-x^2, this gives $\frac{(a+x)^2+(a-x)^2}{(a+x)^2-(a-x)^2}$. 149. Multiply each term of the complex fraction by x^2-a^2: this gets $\frac{2ax}{x^2+a^2}$; the integral part reduced is $\frac{2ax}{x^2-a^2}$; then subtract. 150. Multiply the terms of the complex fraction by abc; it becomes $\frac{3abc-(bc+ac+ab)}{bc+ac-ab}$; then subtract.

Resolution into Series, p. 89.

None of the questions present any difficulty.

VIII. INVOLUTION, p. 97.

1. $15a^2$. 2. a^7. 3. $\frac{6}{a^2}+\frac{b^2}{x^3}$. 4. $bx^2a^{-3}y^2$. 5. $3b^7x^{-6}$. 6. $\frac{a}{x^{n-1}}$. 7. ba^{n-4}. 8. $x^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{x}}$. 9. $bx^{-\frac{5}{4}}$. 10. $x^{-\frac{9}{4}}$. 11. $\frac{1}{2}$. 12. $16a^{12}b^2$. 13. $\sqrt[5]{a^{12}}$. 14. $(a+x)^2$. 15. $4^na^{5n}b^n$. . . . 17. Applying the binomial theorem (Art. 91, p. 93) we have the calculation as on p. 94; we have $n=6$ and $b=-6$ in the general formula; hence $a^n=a^6$; $na^{n-1}b=6a^5\times(-6)= -36a^5$; $\frac{n(n-1)}{1.2}a^{n-2}b^2=\frac{6.5}{2}a^4(-6)^2=540a^4$; $\frac{n(n-1)(n-2)}{1.2.3}\times a^{n-3}b^3=\frac{15.4}{3}a^3\times(-6)^3=-4320a^3$, &c. 18. Will be more readily worked by actual multiplication, using det. coeffs.

19. In the general formula, p. 93, for a put 1, for b put x.

20. Here $\frac{x}{a+b} = x\,(a+b)^{-1}$; to expand $(a+b)^{-1}$, for n in the general formula put -1; then multiply every term by x. The terms are alternately positive and negative. See the same series on p. 85. 21. This is the same as $(a+2b)^{-3}$; expanding by the binomial theorem we get $a^{-3} - 3a^{-4}\,(2b) + 6a^{-5}\,(2b)^2 - 10a^{-6}\,(2b)^3 + 15a^{-7}\,(2b)^4$, &c. . . . 24. Actual multiplication is easier with these low powers. . . . 26. Expand, as in 20 and 21, $(a+x)^{-2}$ and double each term. 27. In the answer to Ex. 19 for n put $n+1$. 28. Substituting the given quantities for n, a, b, in the fifth term of general formula we have $\frac{12.11.10.9}{1.2.3.4}(a^2)^8.\,(-x^2)^4 = 495a^{16}x^8$. 29. Since there are in this case $n+1=17$ terms in the development, the 9th is the middle term. This term is plainly $\frac{n\,(n-1)\;\;.\;\;.\;\;.\;\;(n-7)}{1\,.\,2\,.\;\;.\;\;.\;\;.\;\;.\;\;.\;\;8}\,a^8x^8 = 12{,}870\;a^8x^8$. 30. Here the coefficient of the fifth or $(n-3)\,d$ power in the general formula is $\frac{n\,(n-1)\,(n-2)}{1\,.\,2\,.\,3}$ or $\frac{8\,.\,7\,.\,6}{1\,.\,2\,.\,3}$. 31. The development is obvious: the terms are all positive. 32. 33. 34. The developments are similar to those above given. 35. $x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}$. 36. x^{12}. 37. x^2. 38. $\frac{1}{a^{25}}$. 39. $125a^{\frac{3}{2}}$. 40. $\frac{16}{81}\,.\,a^{4n}$. 41. a^3. 42. $64a^{-6}b^9$. 43. $\frac{1}{4}\,.\,a^4$. 44. $-a^{10}x^{15}$. 45. $\frac{1}{27}\,.\,a^2$. 46. x^{ab}. 47. $a^{mr}b^{nr}c^{pr}$. 48. $\sqrt{x}$. . . . 50. $\frac{a^2}{x^2} - \frac{ab^2}{x^3} + \frac{b}{x} + \frac{1}{a^{m+n}}$. 51. $a^4x^{-8} + 2a^3x^{-2} - abx^{-1} + a^{-(n+q)}$. 52. $\sqrt[n]{a^{3m}}$

53. The development is $(3x)^6 + 6(3x)^5\,.\,2y + \frac{6\,.\,5}{1\,.\,2}(3x)^4.(2y)^2 + \frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}(3x)^3\,.\,(2y)^3 + \frac{6\,.\,5\,.\,4\,.\,3}{1\,.\,2\,.\,3\,.\,4}(3x)^2\,.\,(2y)^4 + \frac{6.5.4.3.2}{1.2.3.4.5} \times (3x)(2y)^5 + \frac{6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6}(2y)^6 =$, &c. 54. 8^{-2}, 4^{-3} or 2^{-6}; 15^{-2}; 5^{-4}; 10^{-3}. 56. $(a+b)^{\frac{5}{4}}$. 57. $\frac{1}{x^n}$. . . . 60. Adopting the third form of the cube of a trinomial (middle of p. 96), we have $[(x^2-2x)+1]^3 = x^6 - 8x^3 + 1 + 3x^4(-2x+1) + 3(2x)^2 \times$

$(x^2+1)+3(x^2-2x)+6x^2(-2x)(1)=x^6-8x^3+1+3x^4-6x^5+12x^2+12x^4+3x^2-6x-12x^3=x^6-6x^5+15x^4-20x^3+15x^2-6x+1$. For the second, taking the form at the foot of the page, we have $(1-3x+3x^2-x^3)^2=1+2(-3x+3x^2-x^3)+9x^2-6x(3x^2-x^3)+9x^4+2(-3x^5)+x^6=1-6x+6x^2-2x^3+9x^2-18x^3+6x^4+9x^4-6x^5+x^6=1-6x+15x^2-20x^3+15x^4-6x^5+x^6$. These results can be more readily obtained by using detached coefficients; but the method has much elegance, and it is useful to practise it.

IX. EVOLUTION.

A. Of Simple Quantities, p. 101.

1. $-2a^2b^5$. 2. $3b^2x$. 3. x^n. 4. x^2. 5. $2a^4$. . . . 7. $-5ab^2$. . . . 11. $\frac{2x^2y}{3a^3b^4}$. 12. $x^{\frac{a}{b}}$. 13. a^{-n}. 14. $a^2b^{-3}x^{2n}\sqrt{5}$. 15. $-3a^{\frac{2}{3}}b^{\frac{1}{9}}x^{-1}$. 16. $\frac{a^{-\frac{1}{3}}}{b^n x^{\frac{1}{3n}}}$. 17. $\sqrt[24]{a}$. 18. $\frac{9a^{-1}(a+b)(2-x)^{-2}}{2cd^{-3}}$. 20. $\frac{3x^4(a+x)^{-2n}}{2b^{-n}c^3}$. 21. x^2. 22. x^3. . . . 24. $(a+x)^3$. 25. $3a^2xy^{-6}$. 26. $13x^{m+1}a^{-n}$. . . . 28. $\frac{4a^3x^4y^{-5}}{5x^2(a+x)}$. 29. $-\frac{7a^4x^{-1}y^{\frac{4}{3}}}{9b^3c^5z^{\frac{1}{3}}}$. 30. $\frac{a^2y^3}{3}$. . . . 32. $\frac{2b^{10}x^{\frac{1}{2}}z}{10a^2}$.

B. Compound Quantities, p. 116.

. . . 4. $4x^4-16x^3+24x^2-16x+4\ (2x^2-4x+2$, Root.

$$
\begin{array}{ll}
2x^2 & 4x^4 \\
2x^2 & -16x^3+24x^2 \\
\hline 4x^2 & -16x^3+16x^2 \\
-4x & \qquad 8x^2-16x+4 \\
\hline 4x^2-4x & \qquad 8x^2-16x+4 \\
-4x & \qquad\qquad 0 \\
\hline 4x^2-8x & \\
+2 & \\
\hline 4x^2-8x+2 &
\end{array}
$$

5. The work is the same as in No. 4. 6. Here $2x^0=2$ is a higher power of x than $\frac{1}{x}=x^{-1}$; ∴ the arrangement is $x+2+\frac{1}{x}$.

$$\begin{array}{l|l}
 & x+2+\frac{1}{x} \quad \left(\sqrt{x}+\sqrt{\frac{1}{x}},\ \text{Root.}\right. \\
\sqrt{x} & x \\
\sqrt{x} & \overline{2+\frac{1}{x}} \\
\overline{2\sqrt{x}+\sqrt{\frac{1}{x}}} & 2+\frac{1}{x} \\
 & \overline{\quad 0 \quad}
\end{array}$$

7.

$$\begin{array}{l|l}
a & a^2+x^2 \left(a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5},\ \text{\&c., Root.}\right. \\
a & a^2 \\
\overline{2a+\frac{x^2}{2a}} & \overline{\quad x^2} \\
\quad \frac{x^2}{2a} & x^2+\frac{x^4}{4a^2} \\
\overline{2a+\frac{x^2}{a}-\frac{x^4}{8a^3}} & \overline{\quad -\frac{x^4}{4a^2}} \\
\quad -\frac{x^4}{8a^3} & -\frac{x^4}{4a^2}-\frac{x^6}{8a^4}+\frac{x^8}{64a^6} \\
\overline{2a+\frac{x^2}{a}-\frac{x^4}{4a^3}+\frac{x^6}{16a^5}} & \overline{\quad \frac{x^6}{8a^4}-\frac{x^8}{64a^6}} \\
 & \frac{x^6}{8a^4}+\frac{x^8}{16a^6}-\frac{x^{10}}{64a^8}+\frac{x^{12}}{256a^{10}} \\
 & \overline{\quad -\frac{5x^8}{64a^6}+\frac{x^{10}}{64a^8}-\frac{x^{12}}{256a^{10}}}
\end{array}$$

8. The work is the same as in 4 and 5.

9.

$$\begin{array}{l|l}
1 & 1-2+\frac{3}{2}-\frac{1}{2}+\frac{1}{16} \\
1 & 1 \\
\hline
2-1 & -2+\frac{3}{2} \\
\quad -1 & -2+1 \\
\hline
2-2+\frac{1}{4} & \quad \frac{1}{2}-\frac{1}{2}+\frac{1}{16} \\
 & \quad \frac{1}{2}-\frac{1}{2}+\frac{1}{16} \\
\hline
 & \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{array}$$

$(1-1+\frac{1}{4}$; or $x^2-x+\frac{1}{4}$, Root.

10. Arrange by powers of c.

$$5ca^{m-2}x^{n+1}+a^mx^n-3.\frac{a}{x}, \text{ Root.}$$

$$\begin{array}{l|l}
 & 25c^2a^{2m-4}x^{2n+2}+10ca^{2m-2}x^{2n+1}-30ca^{m-1}x^n+a^{2m}x^{2n}-6a^{m+1}x^{n-1}+\frac{9a^2}{x^2} \\
5ca^{m-2}x^{n+1} & 25c^2a^{2m-4}x^{2n+2} \\
5ca^{m-2}x^{n+1} & \\
\hline
10ca^{m-2}x^{n+1}+a^mx^n & \quad 10ca^{2m-2}x^{2n+1}-30ca^{m-1}x^n+a^{2m}x^{2n} \\
\qquad\qquad a^mx^n & \quad 10ca^{2m-2}x^{2n+1}+a^{2m}x^{2n} \\
\hline
10ca^{m-2}x^{n+1}+2a^mx^n-3.\frac{a}{x}. & \qquad -30ca^{m-1}x^n-6a^{m+1}x^{n-1}+9.\frac{a^2}{x^2} \\
 & \qquad -30ca^{m-1}x^n-6a^{m+1}x^{n-1}+9.\frac{a^2}{x^2} \\
\hline
 & \qquad\qquad 0
\end{array}$$

11. Root, $1-3+3-1$; *i.e.*, $x^3-3ax^2+3a^2x-1$

$$
\begin{array}{llll}
1 & & & 1-6+15-20+15-6+1 \\
1 & & & 1 \\
\hline
2-3 & & & -6+15 \\
\quad -3 & & & -6+\ 9 \\
\hline
2-6+3 & & & 6-20+15 \\
\qquad +3 & & & 6-18+\ 9 \\
\hline
2-6+6-1 & & & -\ 2+\ 6-6+1 \\
 & & & -\ 2+\ 6-6+1 \\
\hline
 & & & \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}
$$

12. $p+qx+rx^2+sx^3$, Root.

$$
\begin{array}{ll}
p & p^2+2pqx+(2pr+q^2)x^2+2(ps+qr)x^3+(2qs+r^2)x^4+2rsx^5+s^2x^6 \\
p & p^2 \\
\hline
2p+qx & 2pqx+2prx^2+q^2x^2 \\
\quad qx & 2pqx+q^2x^2 \\
\hline
2p+2qx+rx^2 & 2prx^2+2psx^3+2qrx^3+2qsx^4+r^2x^4 \\
\qquad rx^2 & 2prx^2+2qrx^3+r^2x^4 \\
\hline
2p+2qx+2rx^2+sx^3 & 2psx^3+2qsx^4+2rsx^5+s^2x^6 \\
 & 2psx^3+2qsx^4+2rsx^5+s^2x^6 \\
\hline
 & 0
\end{array}
$$

13.

$$
\begin{array}{ll}
 & a^2x^2+px+q^2\ \left(ax+\dfrac{p}{2a}\right. \\
ax & a^2x^2 \\
ax & \\
\hline
2ax+\dfrac{p}{2a} & px+q^2 \\
 & px+\dfrac{p^2}{4a^2} \\
\hline
 & q^2-\dfrac{p^2}{4a^2}
\end{array}
$$

In order, then, that there shall be no remainder, we must

have $q^2 = \frac{p^2}{4a^2}$; hence the condition is that we have $p^2 = 4a^2q^2$, or $p^2x^2 = 4a^2q^2x^2$, that is, a trinomial quantity will be a complete square whenever *four times the product of the first and last terms is equal to the square of the middle term.* Similarly $x^2 \pm px + \frac{p^2}{4}$ is a complete square. (See the subject of Quadratic Equations, Sects. XIV. and XVI.)

14.

$$\begin{array}{lll}
 & & \qquad 2x-5,\ \text{Root.} \\
0 & 0 & 8x^3 - 60x^2 + 150x - 125 \\
\underline{2x} & \underline{4x^2} & \underline{8x^3 \qquad\qquad\qquad\qquad} \\
2x & 4x^2 & \quad -60x^2 + 150x - 125 \\
\underline{2x} & \underline{8x^2} & \underline{\quad -60x^2 + 150x - 125} \\
4x & 12x^2 & \qquad\qquad 0 \\
\underline{2x \qquad} & \underline{\qquad -30x + 25} & \\
6x & 12x^2 - 30x + 25 & \\
\underline{\quad -5} & & \\
6x - 5 & &
\end{array}$$

Here $12x^2$ is the trial divisor, giving -5 for the second part of the root; $6x - 5$ multiplied by this gives the rest of the divisor. It is obvious by inspection that $2x - 5$ is the root.

15. It is obvious, on inspection, that the root is $\frac{ac}{b}x^2 - \frac{b}{c}x$; but it will be a useful exercise on the rule to work the question at length.

$$\frac{ac}{b}x^2 - \frac{b}{c}x, \text{ Root.}$$

0	0	$\frac{a^3c^3}{b^3}x^6 - \frac{3a^2c}{b}x^5 + \frac{3ab}{c}x^4 - \frac{b^3}{c^3}x^3$
$\frac{ac}{b}x^2$	$\frac{a^2c^2}{b^2}x^4$	$\frac{a^3c^3}{b^3}x^6$
$\frac{acx^2}{b}$	$\frac{a^2c^2x^4}{b^2}$	$-\frac{3a^2c}{b}x^5 + \frac{3ab}{c}x^4 - \frac{b^3}{c^3}x^3$
$\frac{acx^2}{b}$	$\frac{2a^2c^2x^4}{b^2}$	$-\frac{3a^2c}{b}x^5 + \frac{3ab}{c}x^4 - \frac{b^3}{c^3}x^3$
$\frac{2acx^2}{b}$	$\frac{3a^2c^2x^4}{b^2}$	0
$\frac{acx^2}{b}$	$-3ax^3 + \frac{b^2}{c^2}x^2$	
$\frac{3acx^2}{b}$	$\frac{3a^2c^2}{b^2}x^4 - 3ax^3 + \frac{b^2}{c^2}x^2.$	
$-\frac{bx}{c}$		
$\frac{3acx^2}{b} - \frac{bx}{c}$		

The trial divisor $\frac{3a^2c^2}{b^2}x^4$ is found by multiplying the sum in the first column $\frac{3ac}{b}x^2$, by the first term of the root; the first term of the remainder divided by this gets $-\frac{b}{c}x$ for the second part of the root, which is annexed in the first column, and the sum multiplied by $-\frac{b}{c}x$, to find the other two terms of the trial divisor.

16. The cube root plainly is $(a+1)^{2n}x+2ca^p$. But we shall find it according to the rule:

$$
\begin{array}{r}
(a+1)^{6n}x^3+6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x+8c^3a^{3p}\ \big|\ (a+1)^{2n}x+2ca^p\text{, Root.}\\
(a+1)^{6n}x^3\\
\hline
6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x+8c^3a^{3p}\\
6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x+8c^3a^{3p}\\
\hline
0
\end{array}
$$

$$
\begin{array}{rr}
(a+1)^{2n}x & (a+1)^{4n}x^2\\
(a+1)^{2n}x & 2(a+1)^{4n}x^2\\
\hline
2(a+1)^{2n}x & 3(a+1)^{4n}x^2\\
(a+1)^{2n}x & +6ca^p(a+1)^{2n}x+4c^2a^{2p}\\
\hline
3(a+1)^{2n}x & 3(a+1)^{4n}x^2+6ca^p(a+1)^{2n}x+4c^2a^{2p}\\
+2ca^p & \\
\hline
3(a+1)^{2n}x+2ca^p &
\end{array}
$$

17.

$$
\begin{array}{r}
x^6-6x^5+15x^4-20x^3+15x^2-6x+1\ \big|\ x^2-2x+1\text{, Root.}\\
x^6\\
\hline
-6x^5+15x^4-20x^3\\
-6x^5+12x^4-\ \ 8x^3\\
\hline
3x^4-12x^3+15x^2-6x+1\\
3x^4-12x^3+15x^2-6x+1\\
\hline
0
\end{array}
$$

$$
\begin{array}{rr}
x^2 & x^4\\
x^2 & 2x^4\\
\hline
2x^2 & 3x^4\\
x^2 & -\ \ 6x^3+\ \ 4x^2\\
\hline
3x^2 & 3x^4-\ \ 6x^3+\ \ 4x^2\\
-2x & -\ \ 6x^3+\ \ 8x^2\\
\hline
3x^2-2x & 3x^4-12x^3+12x^2\\
-2x & 3x^2-6x+1\\
\hline
3x^2-4x & 3x^4-12x^3+15x^2-6x+1\\
-2x & \\
\hline
3x^2-6x & \\
+1 & \\
\hline
3x^2-6x+1 &
\end{array}
$$

Here $2x^2$, the first sum in the first column, is multiplied by x^2, the first term of the root, to find the term $2x^4$ to be added in the second column ; the sum, $3x^4$, is the first trial divisor; with this divisor we find the second term of the root, $-2x$; adding x^2 again in the first column, annexing $-2x$, and multiplying the sum by $-2x$ we get the two other terms of the divisor, $-6x^3+4x^2$; multiplying and subtracting we complete the second step. To find the next divisor add $-2x$ again in the first column, multiply the sum by $-2x$, add the product $-6x^3+8x^2$ in the second column; the sum is the new trial divisor $3x^4-12x^3+12x^2$; which gives 1 for the new term of the root; add $-2x$ again in the first column, and to the sum annex the new term 1, multiply this sum by 1, and add the product, $3x^2-6x+1$ in the second column to complete the divisor; lastly, multiply and subtract, and there is no remainder.

18.

$$\begin{array}{l}
3x^2 \\
3x^2 \\
\hline
6x^2 \\
3x^2 \\
\hline
9x^2 \\
\qquad -2x \\
\hline
9x^2-2x \\
\qquad -2x \\
\hline
9x^2-4x \\
\qquad -2x \\
\hline
9x^2-6x \\
\qquad\qquad +1 \\
\hline
9x^2-6x+1
\end{array}
\qquad
\begin{array}{l}
9x^4 \\
18x^4 \\
\hline
27x^4 \\
\qquad -18x^3+4x^2 \\
\hline
27x^4-18x^3+4x^2 \\
\qquad -18x^3+8x^2 \\
\hline
27x^4-36x^3+12x^2. \\
\qquad\qquad 9x^2-6x+1 \\
\hline
27x^4-36x^3+21x^2-6x+1
\end{array}
\qquad
\begin{array}{l}
\qquad 3x^2-2x+1,\ \text{Root.} \\
\hline
27x^6-54x^5+63x^4-44x^3+21x^2-6x+1 \\
27x^6 \\
\hline
\quad -54x^5+63x^4-44x^3 \\
\quad -54x^5+36x^4-8x^3 \\
\hline
\qquad\qquad 27x^4-36x^3+21x^2-6x+1 \\
\qquad\qquad 27x^4-36x^3+21x^2-6x+1 \\
\hline
\qquad\qquad\qquad 0
\end{array}$$

19.

$$2x^2 + 4ax - 3a^2, \text{ Root.}$$

$$8x^6 + 48ax^5 + 60a^2x^4 - 80a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6$$
$$8x^6$$
$$48ax^5 + 60a^2x^4 - 80a^3x^3$$
$$48ax^5 + 96a^2x^4 + 64a^3x^3$$
$$-36a^2x^4 - 144a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6$$
$$\underline{-36a^2x^4 - 144a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6}$$
$$0$$

$2x^2$	$4x^4$	
$2x^2$	$8x^4$	
$4x^2$	$12x^4$	
$2x^2$		$+ 24ax^3 + 16a^2x^2$
$6x^2$		$12x^4 + 24ax^3 + 16a^2x^2$
$+ \quad 4ax$		$24ax^3 + 32a^2x^2$
$6x^2 + \quad 4ax$		$12x^4 + 48ax^3 + 48a^2x^2$
$4ax$		$- 18a^2x^2 - 36a^3x + 9a^4$
$6x^2 + \quad 8ax$		$12x^4 + 48ax^3 + 30a^2x^2 - 36a^3x + 9a^4$
$4ax$		
$6x^2 + 12ax$		
$- 3a^2$		
$6x^2 + 12ax - 3a^2$		

20.

$$1 + 2x + 3x^2 + 4x^3, \text{ Root.}$$

$$\begin{array}{l}
1 \\
1 \\
\hline
2 \\
1 \\
\hline
3 \\
\quad + 2x \\
\hline
3 + 2x \\
\quad\quad 2x \\
\hline
3 + 4x \\
\quad\quad 2x \\
\hline
3 + 6x \\
\quad\quad + 3x^2 \\
\hline
3 + 6x + 3x^2 \\
\quad\quad\quad 3x^2 \\
\hline
3 + 6x + 6x^2 \\
\quad\quad\quad 3x^2 \\
\hline
3 + 6x + 9x^2 \\
\quad\quad\quad + 4x^3 \\
\hline
3 + 6x + 9x^2 + 4x^3
\end{array}$$

$$\begin{array}{l}
1 \\
2 \\
\hline
3 \\
+ \; 6x + \; 4x^2 \\
\hline
3 + \; 6x + \; 4x^2 \\
\quad\quad 6x + \; 8x^2 \\
\hline
3 + 12x + 12x^2 \\
\quad\quad\quad 9x^2 + 18x^3 + \; 9x^4 \\
\hline
3 + 12x + 21x^2 + 18x^3 + \; 9x^4 \\
\quad\quad\quad 9x^2 + 18x^3 + 18x^4 \\
\hline
3 + 12x + 30x^2 + 36x^3 + 27x^4 \\
\quad\quad\quad\quad 12x^3 + 24x^4 + 36x^5 + 16x^6 \\
\hline
3 + 12x + 30x^2 + 48x^3 + 51x^4 + 36x^5 + 16x^6
\end{array}$$

$$\begin{array}{l}
1 + 6x + 21x^2 + 56x^3 + 111x^4 + 174x^5 + 219x^6 + 204x^7 + 144x^8 + 64x^9 \\
1 \\
\hline
\quad 6x + 21x^2 + 56x^3 \\
\quad 6x + 12x^2 + \; 8x^3 \\
\hline
\quad\quad 9x^2 + 48x^3 + 111x^4 + 174x^5 + 219x^6 \\
\quad\quad 9x^2 + 36x^3 + \; 63x^4 + \; 54x^5 + \; 27x^6 \\
\hline
\quad\quad\quad 12x^3 + \; 48x^4 + 120x^5 + 192x^6 + 204x^7 + 144x^8 + 64x^9 \\
\quad\quad\quad 12x^3 + \; 48x^4 + 120x^5 + 192x^6 + 204x^7 + 144x^8 + 64x^9 \\
\hline
\quad\quad\quad\quad\quad 0
\end{array}$$

21. Ordering the terms by the powers of a, the work, without the letters, is as follows:

		$1-2$, *i.e.*, $a^m - 2ax^n$, Root.
		$1-6+12-8$
1	1	1
1	2	$-6+12-8$
2	3	$-6+12-8$
1	$-6+4$	0
3	$3-6+4$	
-2		
$3-2$		

22.

		$3+2+7$, *i.e.*, $3x^2+2x+7$, Root.
0	0	$27+54+225+260+525+294+343$
3	9	27
3	9	$54+225+260$
3	18	$54+36+8$
6	27	$189+252+525+294+343$
3	$+18+4$	$189+252+525+294+343$
9	$27+18+4$	0
$+2$	$18+8$	
$9+2$	$27+36+12$	
2	$63+42+49$	
$9+4$	$27+36+75+42+49$	
2		
$9+6$		
$+7$		
$9+6+7$		

23.

$$\begin{array}{l}
1\\
1\\
\hline 2\\
1\\
\hline 3\\
\quad -1\\
\hline 3-1\\
\quad -1\\
\hline 3-2\\
\quad -1\\
\hline 3-3\\
\qquad +2\\
\hline 3-3+2
\end{array}
\qquad
\begin{array}{l}
1\\
2\\
\hline 3\\
\quad -3+1\\
\hline 3-3+1\\
\quad -3+2\\
\hline 3-6+3\\
\qquad\quad 6-6+4\\
\hline 3-6+9-6+4
\end{array}
\qquad
\begin{array}{l}
\qquad\quad 1-1+2 \text{ Root.}\\
\hline 1-3+9-13+18-12+8\\
1\\
\hline -3+9-13\\
-3+3-\ 1\\
\hline \qquad 6-12+18-12+8\\
\qquad 6-12+18-12+8\\
\hline \qquad\qquad 0
\end{array}$$

24.

$$\begin{array}{l}
ax\\
ax\\
\hline 2ax\\
ax\\
\hline 3ax\\
\quad +\dfrac{b}{3a^2}\\
\hline 3ax+\dfrac{b}{3a^2}
\end{array}
\qquad
\begin{array}{l}
a^2x^2\\
2a^2x^2\\
\hline 3a^2x^2\\
\quad +\dfrac{b}{a}x+\dfrac{b^2}{9a^4}\\
\hline 3a^2x^2+\dfrac{b}{a}x+\dfrac{b^2}{9a^4}
\end{array}
\qquad
\begin{array}{l}
ax+\dfrac{b}{3a^2}, \text{ Root.}\\
\hline a^3x^3+bx^2+cx+d\\
a^3x^3\\
\hline bx^2+cx+d\\
bx^2+\dfrac{b^2}{3a^3}x+\dfrac{b^3}{27a^6}\\
\hline cx-\dfrac{b^2}{3a^3}x+d-\dfrac{b^3}{27a^6}
\end{array}$$

Hence, in order that there may be no remainder, and the given quantity a perfect cube, we must have $c=\dfrac{b^2}{3a^3}$, $d=\dfrac{b^3}{27a^6}$; and these are the required values of c and d.

25. Let the complete cube be represented by $(ax+b)^3$, then this expression is identical with the given one, mx^3+nx^2+px+q; so that comparing the terms of the development of the former with those of the latter, we have $m=a^3$, $n=3a^2b$, $p=3ab^2$, $q=b^3$, hence $\dfrac{n^3}{p^3}=\dfrac{27a^6b^3}{27a^3b^6}=\dfrac{a^3}{b^3}=\dfrac{m}{q}$; $\therefore\ mp^3=qn^3$. Also, $n^2=9a^4b^2=9a^4\times\dfrac{p}{3a}=3a^3p=3mp$.

C. By the Binomial Theorem, p. 118.

26. Here developing by the binomial theorem (Art. 91, p. 93), we have $(b^2+x)^{\frac12} = (b^2)^{\frac12} + \frac12(b^2)^{-\frac12}x + \frac{\frac12(-\frac12)}{1\,.\,2}(b^2)^{-\frac32}x^2 + \frac{\frac12(-\frac12)(-\frac32)}{1\,.\,2\,.\,3}(b^2)^{-\frac52}x^3 + \frac{\frac12(-\frac12)(-\frac32)(-\frac52)}{1\,.\,2\,.\,3\,.\,4}(b^2)^{-\frac72}x^4 +$ &c. $= b + \frac{x}{2b} - \frac{x^2}{8b^3} + \frac{x^3}{16b^5} - \frac{5x^4}{128b^7} + \frac{7x^5}{256b^9} - \frac{21x^6}{1024b^{11}}$&c. $= b + \frac{x}{2b} - \frac{x^2}{2.4b^3} + \frac{3x^3}{2\,.\,4\,.\,6b^5} - \frac{3\,.\,5x^4}{2\,.\,4\,.\,6\,.\,8b^7} + \frac{3\,.\,5\,.\,7x^5}{2.4.6.8.10b^9} - \frac{3\,.\,5\,.\,7\,.\,9x^6}{2.4.6.8.10.12b^{11}} +$ &c. $= b + \frac{x}{2b} - \frac{x^2}{2^3b^3} + \frac{x^3}{2^4b^5} - \frac{5x^4}{2^7b^7} + \frac{7x^5}{2^8b^9} - \frac{3\,.\,7x^6}{2^{10}b^{11}}$ &c.

27. Here $(b^2+x)^{-\frac12} = (b^2)^{-\frac12} - \frac12(b^2)^{-\frac32}x + \frac{-\frac12(-\frac32)}{1\,.\,2}(b^2)^{-\frac52}x^2 - \frac{-\frac12(-\frac32)(-\frac52)}{1\,.\,2\,.\,3}(b^2)^{-\frac72}x^3$ &c. $= \frac1b - \frac{x}{2b^3} + \frac{3x^2}{2\,.\,4b^5} - \frac{3\,.\,5x^3}{2\,.\,4\,.\,6b^7} + \frac{3\,.\,5\,.\,7x^4}{2\,.\,4\,.\,6\,.\,8b^9} -$ &c.

28. $(1+1)^{\frac12} = 1 + \frac12 - \frac18 + \frac1{16} - \frac5{128} + \frac7{256} -$ &c. $= 1 + \frac12 - \frac1{2.4} + \frac{1\,.\,3}{2\,.\,4\,.\,6} - \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8} + \frac{1\,.\,3\,.\,5\,.\,7}{2\,.\,4\,.\,6\,.\,8\,.\,10} - \frac{1\,.\,3\,.\,5\,.\,7\,.\,9}{2\,.\,4.6\,.\,8\,.\,10.12} +$ &c.

29. The coefficients are the same as in the last example.

30. Substitute 1 for a, and $\frac1n$ for n, then $(1+x)^{\frac1n} = 1 + \frac1n x + \frac{\frac1n(\frac1n-1)}{1\,.\,2}x^2 + \frac{\frac1n(\frac1n-1)(\frac1n-2)}{1\,.\,2\,.\,3}x^3 + \frac{\frac1n(\frac1n-1)(\frac1n-2)(\frac1n-3)}{1\,.\,2\,.\,3\,.\,4}x^4$ &c. $= 1 + \frac1n x - \frac{n-1}{2n^2}x^2 + \frac{(n-1)(2n-1)}{2\,.\,3n^3}x^3 - \frac{(n-1)(2n-1)(3n-1)}{2\,.\,3\,.\,4n^4}x^4 +$ &c.

31. $(a+x)^{\frac23} = a^{\frac23} + \frac23 a^{-\frac13}x + \frac{\frac23(-\frac13)}{1\,.\,2}a^{-\frac43}x^2 + \frac{\frac23(-\frac13)(-\frac43)}{1\,.\,2\,.\,3}a^{-\frac73}x^3 + \frac{\frac23(-\frac13)(-\frac43)(-\frac73)}{1\,.\,2\,.\,3\,.\,4}a^{-\frac{10}3}x^4 +$ &c. $= a^{\frac23} + \frac{2x}{3a^{\frac13}} - \frac{x^2}{9a^{\frac43}} + \frac{4x^3}{9^2a^{\frac73}} - \frac{4\,.\,7x^4}{9^2\,.\,12a^{\frac{10}3}} + \frac{4\,.\,7\,.\,10x^5}{9^2.12\,.\,15a^{\frac{13}3}} - \frac{4\,.\,7\,.\,10\,.\,13x^6}{9^2\,.\,12\,.\,15\,.\,18a^{\frac{16}3}} +$ &c.

32. The work is the same as that of Ex. 2, p. 116.

33. The coefficients are the same as in Ex. 27; the powers of the letters are $(c^2)^{-\frac{1}{2}}$, $(c^2)^{-\frac{3}{2}}x^2$, $(c^2)^{-\frac{5}{2}}x^4$ &c., or $\frac{1}{(c^2)^{\frac{1}{2}}} = \frac{1}{c}$, $\frac{x^2}{c^3}$, $\frac{x^4}{c^5}$, &c. $\therefore a\,(c^2+x^2)^{-\frac{1}{2}} = a(\frac{1}{c} - \frac{x^2}{2c^3} + \frac{3x^4}{8c^5} - \frac{5x^6}{16c^7} + \frac{5\,.\,7x^8}{128c^9} -$ &c.$) = a(\frac{1}{c} - \frac{x^2}{2c^3} + \frac{3x^4}{2\,.\,4c^5} - \frac{3\,.\,5x^6}{2\,.\,4\,.\,6c^7} + \frac{3\,.\,5\,.\,7x^8}{2\,.\,4\,.\,6\,.\,8c^9} -$ &c.$)$

34. Development as in Ex. 33, but terms positive.

35. Here $+\frac{-\frac{1}{n}(-\frac{1}{n}-1)}{1\,.\,2} = +\frac{-n-1}{1\,.\,2n} \times (-\frac{1}{n}) = -\frac{n+1}{1\,.\,2n} \times (-\frac{1}{n}) = \frac{n+1}{2n^2}$; $+\frac{n+1}{2n^2}\cdot\left(\frac{-\frac{1}{n}-2}{3}\right) = +\frac{n+1}{2n^2}\cdot\left(\frac{-2n-1}{3n}\right) = +\frac{n+1}{2n^2}\cdot\left(-\frac{2n+1}{3n}\right) = -\frac{(n+1)(2n+1)}{2\,.\,3n^3}$; similarly the fifth coefficient is $+\frac{(n+1)(2n+1)(3n+1)}{2\,.\,3\,.\,4n^4}$ &c.

36. Here $(1+1)^{\frac{1}{5}} = 1 + \frac{1}{5} + \frac{\frac{1}{5}(-\frac{4}{5})}{1\,.\,2} + \frac{2}{25}\frac{(\frac{1}{5}-2)}{3} + \frac{6}{125}\frac{(\frac{1}{5}-3)}{4}$ &c. $= 1 + \frac{1}{5} - \frac{2}{25} + \frac{6}{125} - \frac{21}{625} +$ &c.

37. The coefficients are here as in Ex. 26, 33, &c. 38. $(a+x)^{\frac{4}{3}} = a^{\frac{4}{3}} + \frac{4}{3}a^{\frac{1}{3}}x + \frac{\frac{4}{3}(\frac{1}{3})}{1\,.\,2}a^{-\frac{2}{3}}x^2 + \frac{\frac{4}{3}(\frac{1}{3})(-\frac{2}{3})}{1\,.\,2\,.\,3}a^{-\frac{5}{3}}x^3$ &c. $= a^{\frac{4}{3}} + \frac{4a^{\frac{1}{3}}x}{3} + \frac{2x^2}{9a^{\frac{2}{3}}} - \frac{4x^3}{81a^{\frac{5}{3}}} + \frac{5x^4}{243a^{\frac{8}{3}}}$ &c. $= a^{\frac{4}{3}} + \frac{1\,.\,4a^{\frac{1}{3}}}{3}x + \frac{1\,.\,4x^2}{3\,.\,6a^{\frac{2}{3}}} - \frac{1\,.\,2\,.\,4x^3}{3\,.\,6\,.\,9a^{\frac{5}{3}}} + \frac{1\,.\,2\,.\,4\,.\,5x^4}{3\,.\,6\,.\,9\,.\,12a^{\frac{8}{3}}} -$ &c. $= a^{\frac{1}{3}}\{a + \frac{1\,.\,4\,.\,x}{3} + \frac{1\,.\,4\,.\,x^2}{3\,.\,6\,.\,a} - \frac{1\,.\,2\,.\,4\,.\,x^3}{3\,.\,6\,.\,9\,.\,a^2} + \frac{1\,.\,2\,.\,4\,.\,5\,.\,x^4}{3\,.\,6\,.\,9\,.\,12\,.\,a^3} -$ &c.$\}$

X. SURDS, p. 128.

. . . 10. Here $\frac{2}{3} \div \frac{5}{6} = \frac{4}{5}$, the required index. . . . 12. $a = (a^{\frac{q}{p}})^{\frac{p}{q}} \therefore$ &c. 13. $9^{\frac{3}{2}} = (3^2)^{\frac{3}{2}} = 3^3 = 27$. 14. $\sqrt{a} = a^{\frac{1}{2}}$; $\frac{1}{2} \div \frac{1}{3} = \frac{3}{2}$, the index required. . . . 16. $8a^3$ is the common factor. . . . 26. The factors of the num. are $(a-2x)^2$ and ax. . . . 28. Multiply both terms of the first fraction by $\sqrt{c}$, and both terms of the second by $\sqrt{d}$, giving $\left(\frac{a^3b^2c}{c^2d^2} - \frac{a^2b^3d}{c^2d^2}\right)^{\frac{1}{2}}$; then re-

move the common factor $\frac{a^2b^2}{c^2d^2}$ . . . 30. The factors are $(a+b)^2$ and $3c$. 31. The factors are $(y-1)^2$ and $2n$. . . 40. $\frac{1}{3}\sqrt[3]{\frac{16}{81}}=\frac{1}{3}\sqrt[3]{\frac{144}{729}}=\frac{1}{3}\cdot\frac{1}{9}\sqrt[3]{(8\times 18)}=\frac{2}{27}\sqrt[3]{18}$. . . . 53. $\sqrt{\frac{25}{3}}=5\sqrt{\frac{1}{3}}=\frac{5}{3}\sqrt{3}$; $\sqrt{\frac{4}{27}}=2\sqrt{\frac{1}{27}}=\frac{2}{9}\sqrt{3}$; $\sqrt{\frac{9}{48}}=\sqrt{\frac{9}{16}}\sqrt{\frac{1}{3}}=\frac{3}{4}\sqrt{\frac{1}{3}}=\frac{1}{4}\sqrt{3}$, $\therefore$ &c. . . . 61. $(4a^2)^{\frac{1}{6}}=(2a)^{\frac{1}{3}}$; $(8a^3)^{\frac{1}{9}}=(2a)^{\frac{1}{3}}$, &c. . . . 66. $486=243\times 2=3^5\times 2$, &c. . . .

77. $\frac{b}{c}\sqrt{ab}+\frac{(a-2b)\sqrt{ab}}{2c}=\frac{2b}{2c}\sqrt{ab}+\frac{(a-2b)\sqrt{ab}}{2c}=\sqrt{ab}\times\left(\frac{a-2b+2b}{2c}\right)=\frac{a}{2c}\sqrt{ab}$ . . . 100, 101. The detached coeffs. may here be used with advantage. . . . 115. $72^{\frac{1}{6}}\div 3^{\frac{1}{3}}=72^{\frac{1}{6}}\div 3^{\frac{2}{6}}=72^{\frac{1}{6}}\div 9^{\frac{1}{6}}=8^{\frac{1}{6}}=(2^3)^{\frac{1}{6}}=\sqrt{2}$. . . . 119. $(n^2-m^2)^{\frac{1}{2}}=\sqrt{(n-m)}\sqrt{(n+m)}$; and $n-m=\sqrt{(n-m)}\sqrt{(n-m)}$ $\therefore$ $\sqrt{(n-m)}$ divides both terms. . . . 122. $a^{\frac{1}{5}}\div a^{\frac{1}{6}}=a^{\frac{1}{5}-\frac{1}{6}}=a^{\frac{1}{30}}$. . . . 126. $2=32^{\frac{1}{5}}$, $\therefore$ &c. . . . 129. The quotient has the same form as in Ex. 78 in Division; and the detached coefficients may be employed; the indices of a diminish, and those of x increase by $\frac{1}{2}$ in each term. . . . 137. Here $2(a^2b^3)^{\frac{1}{4}}=2a^{\frac{1}{2}}b^{\frac{3}{4}}$; $a^2(a^3b^2)^{\frac{1}{6}}=a^{\frac{15}{6}}b^{\frac{2}{6}}=a^{\frac{5}{2}}b^{\frac{1}{3}}$; the work $\therefore$ is as follows:

$$a^{\frac{1}{2}}-b^{\frac{1}{3}})\,a^3-2a^{\frac{1}{2}}b^{\frac{3}{4}}-a^{\frac{5}{2}}b^{\frac{1}{3}}+2b^{\frac{13}{12}}\,(a^{\frac{5}{2}}-2b^{\frac{3}{4}};\text{ or, } a^2\sqrt{a}-2\sqrt[4]{b^3}$$
$$\begin{array}{l}
a^3-a^{\frac{5}{2}}b^{\frac{1}{3}} \\
\hline
-2a^{\frac{1}{2}}b^{\frac{3}{4}}+2b^{\frac{13}{12}} \\
-2a^{\frac{1}{2}}b^{\frac{3}{4}}+2b^{\frac{13}{12}} \\
\hline
0
\end{array}$$

138. It will at once be seen, without performing the division, that the other factor is $a+\sqrt{b}$.

140. $$a^{\frac{6}{5}}-2a^{\frac{4}{5}}+3a^{\frac{2}{5}}-4)\,4a^{\frac{9}{5}}-9a^{\frac{7}{5}}+14a-19a^{\frac{3}{5}}+4a^{\frac{1}{5}}\,(4a^{\frac{3}{5}}-a^{\frac{1}{5}}$$
$$\begin{array}{l}
4a^{\frac{9}{5}}-8a^{\frac{7}{5}}+12a-16a^{\frac{3}{5}} \\
\hline
-a^{\frac{7}{5}}+2a-3a^{\frac{3}{5}}+4a^{\frac{1}{5}} \\
-a^{\frac{7}{5}}+2a-3a^{\frac{3}{5}}+4a^{\frac{1}{5}} \\
\hline
0
\end{array}$$

141. Assume $\sqrt{x}+\sqrt{y}=\sqrt{(7+4\sqrt{3})}$, then squaring we have $x+y+2\sqrt{xy}=7+4\sqrt{3}$; $\therefore$ (Art. 102), $x+y=7$, $2\sqrt{xy}=4\sqrt{3}$; $\therefore +\sqrt{xy}=2\sqrt{3}$, $\therefore$ $xy=12$; also, $x^2+2xy+y^2=49$, and $4xy=48$; $\therefore$ $(x^2+2xy+y^2)-4xy=x^2-2xy+y^2=1$; and $\therefore$ $x-y=1$; but $x+y=7$, $\therefore$ $x=4$ and $y=3$. Substituting these values in the expression $\sqrt{x}+\sqrt{y}$ we have $\sqrt{x}+\sqrt{y}=2+\sqrt{3}$, the root required. The same result will be obtained by substituting in the final formula, p. 124, 7 for a and $4\sqrt{3}$ for b, we thus get

$$\sqrt{\frac{7+\sqrt{(49-48)}}{2}}+\sqrt{\frac{7-\sqrt{(49-48)}}{2}}=\frac{\sqrt{8}}{\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{2}}=2+\sqrt{3}.$$

. . . 143. Assume $\sqrt{x}-\sqrt{y}=\sqrt{(17-12\sqrt{2})}$; $\therefore$ squaring $x+y=17$, $2\sqrt{xy}=12\sqrt{2}$; $\therefore$ $\sqrt{xy}=6\sqrt{2}$, and $xy=72$; $\therefore$ $4xy=288$. Also, $x^2+2xy+y^2=289$, $\therefore$ $x^2+2xy+y^2-4xy=289-288=1$, and $x-y=1$; but $x+y=17$; $\therefore$ $x=9$, $y=8$. Hence, $\sqrt{x}-\sqrt{y}=3-\sqrt{8}=3-2\sqrt{2}$. . . . 145. Here we find in the usual way $x+y=\frac{3}{2}$, $4xy=2$ $\therefore$ $x^2-2xy+y^2=\frac{9}{4}-2=\frac{1}{4}$, and $x-y=\frac{1}{2}$ but $x+y=\frac{3}{2}$ $\therefore$ $x=1$, $y=\frac{1}{2}$; so that the root is $1+\sqrt{\frac{1}{2}}=1+\frac{1}{2}\sqrt{2}$. . . .

147. Here (Art. 114, p. 125), $\sqrt{32}-\sqrt{24}=\sqrt{2}(4-\sqrt{12})$ $\therefore$ $\sqrt{(\sqrt{32}-\sqrt{24})}=\sqrt[4]{2}\sqrt{(4-\sqrt{12})}$. Now, $\sqrt{(4-\sqrt{12})}$ found as in the preceding Exs. is $\sqrt{3}-1$; $\therefore$ $\sqrt[4]{2}\sqrt{(4-\sqrt{12})}=\sqrt[4]{2}(\sqrt{3}-1)=\sqrt[4]{2}(\sqrt[4]{9}-1)=\sqrt[4]{18}-\sqrt[4]{2}$, for $\sqrt{3}=\sqrt[4]{9}$. The given surd is of the form $\sqrt{a^2c}+\sqrt{b}=\sqrt{c}\left(a+\sqrt{\frac{b}{c}}\right)$; and if $a^2-\frac{b}{c}$ be a complete square, then $\sqrt{\left(a+\sqrt{\frac{b}{c}}\right)}$ may be expressed in the form $\sqrt{x}+\sqrt{y}$; that is, the square root of $\sqrt{a^2c}+\sqrt{b}$ will be $\sqrt[4]{c}(\sqrt{x}+\sqrt{y})$. In this case $a^2=16$, $c=2$, $b=24$ $\therefore$ $a^2-\frac{b}{c}=16-12=4$ a perfect square. It is obvious that if in the general formula $a^2-\frac{b}{c}$ be not a perfect square, the surd will not be reducible to a form more simple than the given one, $\sqrt{(a+\sqrt{b})}$. The same result will be obtained by substitution in the general formula p. 124; thus $\sqrt{(\sqrt{32}-\sqrt{24})}=$

$$\sqrt{\frac{\sqrt{32}+\sqrt{(32-24)}}{2}}-\sqrt{\frac{\sqrt{32}-\sqrt{(32-24)}}{2}}=$$

$$\sqrt{\frac{\sqrt{32}+\sqrt{8}}{2}}-\sqrt{\frac{\sqrt{32}-\sqrt{8}}{2}}=\sqrt{(\sqrt{16}+\sqrt{4})}-\sqrt{(\sqrt{16}-\sqrt{4})}=\sqrt{6}-\sqrt{2}.$$ Now, $\sqrt{6}=\sqrt{2}\times\sqrt{3}$; $\therefore$ $\sqrt{6}-\sqrt{2}=$

$\sqrt{2}(\sqrt{3}-1)$, and the square root of this is as before, $\sqrt[4]{2}(\sqrt[4]{9}-1)=\sqrt[4]{18}-\sqrt[4]{2}$.

149. For a put 94, and for $\sqrt{b}$, $42\sqrt{5}$, in the general formula, and we have $x=\dfrac{94+\sqrt{(8836-8820)}}{2}$, $y=\dfrac{94-\sqrt{(8836-8820)}}{2}$; or, $x=49$, $y=45$, $\therefore\ \sqrt{x}-\sqrt{y}=7-3\sqrt{5}$. 150. Here $3\sqrt{5}+\sqrt{40}=\sqrt{5}(3+\sqrt{8})$ $\therefore\ \sqrt{\{\sqrt{5}(3+\sqrt{8})\}}=\sqrt[4]{5}\sqrt{(3+\sqrt{8})}$. Taking the square root of $3+\sqrt{8}$ as in the preceding Ex. we find $x+y=3$, and $2\sqrt{xy}=\sqrt{8}$ $\therefore\ 4xy=8$; also $x^2+2xy+y^2=9$ $\therefore\ (x+y)^2-4xy=1$ and $x-y=1$, but $x+y=3$ $\therefore\ x=2$, and $y=1$. Hence $\sqrt[4]{5}\sqrt{(3+\sqrt{8})}=\sqrt[4]{5}(\sqrt{2}+1)=\sqrt[4]{5}(\sqrt[4]{4}+1)=\sqrt[4]{20}+\sqrt[4]{5}$. 151. $\sqrt{27}+2\sqrt{6}=3\sqrt{3}+2\sqrt{2}\sqrt{3}=\sqrt{3}(3+2\sqrt{2})$, $\therefore\ \sqrt[4]{3}\sqrt{(3+2\sqrt{2})}=\sqrt[4]{3}(\sqrt{2}+1)$, the latter factor being found as in preceding Ex. Now $\sqrt[2]{2}=\sqrt[4]{4}$ $\therefore\ \sqrt[4]{3}(\sqrt{2}+1)=\sqrt[4]{12}+\sqrt[4]{3}$, the root required. 152. Substituting these values in the general formula we have,

$$\begin{aligned}\sqrt{(4\sqrt{3}+6)}&=\sqrt{\frac{4\sqrt{3}+\sqrt{(48-36)}}{2}}+\sqrt{\frac{4\sqrt{3}-\sqrt{(48-36)}}{2}}\\&=\sqrt{\frac{4\sqrt{3}+\sqrt{12}}{2}}+\sqrt{\frac{4\sqrt{3}-\sqrt{12}}{2}}\\&=\sqrt{\frac{4\sqrt{3}+2\sqrt{3}}{2}}+\sqrt{\frac{4\sqrt{3}-2\sqrt{3}}{2}}\\&=\sqrt{(3\sqrt{3})}+\sqrt{(\sqrt{3})}\\&=\sqrt{(\sqrt{9}\sqrt{3})}+\sqrt{(\sqrt{3})}=\sqrt[4]{27}+\sqrt[4]{3}.\end{aligned}$$

153. Assume $\sqrt{(8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$ $\therefore 8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}=x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{xz}$ put $2\sqrt{xy}=2\sqrt{2}$, $2\sqrt{yz}=2\sqrt{5}$, $2\sqrt{xz}=2\sqrt{10}$; hence $\sqrt{(xy)}\times\sqrt{(yz)}=\sqrt{2}\times\sqrt{5}=\sqrt{10}$; also $\sqrt{xz}=\sqrt{10}$, now $\sqrt{(xy)}\times\sqrt{(yz)}=y\sqrt{xz}=y\sqrt{10}$; $\therefore\ y\sqrt{10}=\sqrt{10}$, and $y=1$, hence we easily find $x=2$, $z=5$. Now (Art. 115) these values satisfy the condition $x+y+z=8$; the required root therefore is $\sqrt{2}+\sqrt{1}+\sqrt{5}$, or $1+\sqrt{2}+\sqrt{5}$. The square of this is the given quantity. See formula p. 96. 154. Proceeding as in Art. 116 we have $\sqrt[3]{(100-108)}=x^2-y$, that is $-2=x^2-y$ and $y=x^2+2$; if now we cube the quantities $\sqrt[3]{(10+\sqrt{108})}$ and $x+\sqrt{y}$ we shall have the rational parts equal (Art. 102), that is $10=x^3+3xy=x^3+3x(x^2+2)$, or $4x^3+6x=10$; an expression which is satisfied by the value $x=1$; this gives $y=3$, and the required cube root therefore is

$1+\sqrt{3}$. 155. We have here as in the last Ex. $\sqrt[3]{(256-320)}=x^2-y$.·. $\sqrt[3]{-64}=x^2-y$, *i.e.*, $-4=x^2-y$, and $y=x^2+4$. Also $16=x^3+3xy$ (Art. 102) .·. $16=x^3+3x(x^2+4)=4x^3+12x$. Now this expression $4x^3+12x=16$ is satisfied by the value $x=1$; but $16=x^3+3xy$.·. $16=1+3y$; .·. $15=3y$ and $y=5$; and hence the required cube root is $1+\sqrt{5}$; since according to our assumption $\sqrt[3]{(16+8\sqrt{5})}=x+\sqrt{y}$. 156.... ... 160. The multipliers are $a-\sqrt{b}$; $-5+\sqrt{\frac{3}{4}}$; $\sqrt{a}-\sqrt{b}$; $a\sqrt{b}-c\sqrt{d}$; $9-2\sqrt{10}$. 161. Multiply both terms by $\sqrt{3}-2$, then $\frac{\sqrt{3}-2}{-1}=2-\sqrt{3}$. 162. Multiply both terms by $2+\sqrt{2}$; this gives $(4+3\sqrt{2})\div(4-2)=2+\frac{3}{2}\sqrt{2}$. 165. Multiply both terms by $\sqrt{2}-3\sqrt{\frac{1}{2}}$; this gets for the numerator $\frac{1}{2}\sqrt{1}-\frac{3}{2}\times\frac{1}{2}=\frac{1}{2}-\frac{3}{4}=-\frac{1}{4}$; for the denomr. $2-9\times\frac{1}{2}=2-4\frac{1}{2}=-2\frac{1}{2}$; then $(-\frac{1}{4})\div(-\frac{5}{2})=\frac{1}{10}$. Or thus, $\frac{1}{2}\sqrt{\frac{1}{2}}=\frac{1}{4}\sqrt{2}$, and $3\sqrt{\frac{1}{2}}=\frac{3}{2}\sqrt{2}$; .·. the expression becomes $\frac{\frac{1}{4}\sqrt{2}}{\sqrt{2}+\frac{3}{2}\sqrt{2}}$ multiply both terms by 4; then $\frac{\sqrt{2}}{4\sqrt{2}+6\sqrt{2}}=\frac{1}{4+6}=\frac{1}{10}$. 163, 164, 166. The multipliers are $\sqrt{5}+\sqrt{3}$; $b-\sqrt{c}$ and $\sqrt{a}+\sqrt{b}$. 167. Here (Art. 117) $a=5$, $b=2$, $n=3$; .·. $\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}b}+$ &c. $=\sqrt[3]{25}+\sqrt[3]{10}+\sqrt[3]{4}$ which is the multiplier required. Multiplying, the denomr. becomes $\sqrt[3]{125}-\sqrt[3]{8}=5-2=3$; .·. &c. 168. Multiply both terms by $\sqrt{(1+x)}$; then

$$\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}+\frac{1}{\sqrt{(1-x)}}}=\frac{\sqrt{(1-x^2)}+1}{\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x)}}}=(\sqrt{1-x^2}+1)\div$$

$$\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x)}}=(\sqrt{1-x^2}+1)\times\frac{\sqrt{(1-x)}}{\sqrt{(1-x^2)}+1}=\sqrt{(1-x)}\text{; or}$$

thus, reduce both terms in the usual way, then $\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1+x)}}\div$ $\frac{\sqrt{(1-x^2)}+1}{\sqrt{(1-x^2)}}=\frac{\sqrt{(1-x^2)}}{\sqrt{(1+x)}}=\sqrt{(1-x)}$. 169. Multiply both terms by the numerator; then $\frac{x+a+x-a+2\sqrt{(x^2-a^2)}}{x+a-(x-a)}=$ $\frac{2x+2\sqrt{(x^2-a^2)}}{2a}=\frac{x+\sqrt{(x^2-a^2)}}{a}=\frac{x}{a}+\sqrt{\frac{x^2-a^2}{a^2}}=\frac{x}{a}+$ $\sqrt{\frac{x^2}{a^2}-1}$. 170. Multiply both terms by the denominator

with middle sign changed, and divide by a; we thus get

$$\frac{a(a+x)+a\sqrt{(ax+x^2)}}{a+x-x}=a+x+\sqrt{(ax+x^2)}.$$

XI. ARITHMETIC OF IMAGINARIES, p. 135.

Ex. 9. The factors are $2\sqrt{3}\sqrt{-1}$ and $3\sqrt{2}\sqrt{-1}$; hence the product is $6\sqrt{6}\times(-1)=-6\sqrt{6}$.

10. The work is as follows:

$$\begin{array}{l}
a+b\sqrt{-1} \\
a+b\sqrt{-1} \\
\hline
a^2+ab\sqrt{-1} \\
\qquad ab\sqrt{-1}-b^2 \\
\hline
a^2+2ab\sqrt{-1}-b^2 \\
a+b\sqrt{-1} \\
\hline
a^3+2a^2b\sqrt{-1}-ab^2 \\
\qquad a^2b\sqrt{-1}-2ab^2-b^3\sqrt{-1} \\
\hline
a^3+3a^2b\sqrt{-1}-3ab^2-b^3\sqrt{-1}.
\end{array}$$

15.

$$\begin{array}{l}
-\frac{1}{2}+\frac{1}{2}\sqrt{-3} \\
-\frac{1}{2}+\frac{1}{2}\sqrt{-3} \\
\hline
\frac{1}{4}-\frac{1}{4}\sqrt{-3} \\
\qquad -\frac{1}{4}\sqrt{-3}-\frac{3}{4} \\
\hline
-\frac{1}{2}-\frac{1}{2}\sqrt{-3} \\
-\frac{1}{2}+\frac{1}{2}\sqrt{-3} \\
\hline
\frac{1}{4}+\frac{1}{4}\sqrt{-3} \\
\qquad -\frac{1}{4}\sqrt{-3}+\frac{3}{4} \\
\hline
\qquad 1
\end{array}$$

In cubing the other expression some of the signs will be different, but the result is the same. 16. Here $\frac{6\sqrt{-3}}{2\sqrt{-4}}=3\sqrt{\frac{3}{4}}=\frac{3}{2}\sqrt{3}$.

17. Multiply both terms by the denom. with middle sign changed, *i.e.*, by $1+\sqrt{-1}$; this gives $\frac{2\sqrt{-1}}{2}=\sqrt{-1}$.

18. Here 100 is made up of the square numbers 36 and 64; the factors are, $\sqrt{36}+\sqrt{64}\sqrt{-1}$, and $\sqrt{36}-\sqrt{64}\sqrt{-1}$; or

$6+8\sqrt{-1}$ and $6-8\sqrt{-1}$, &c. 19. The product of the imaginary factors is x^2+a^2, $\therefore$ &c. 20. Multiply both terms by $a+b\sqrt{-1}$. 21. The product of these factors is $576+49=625=25^2=24^2+7^2$. Also their conjugates are the imaginary factors of the same quantity. 22. Here $\dfrac{3-\sqrt{-2}}{2+3\sqrt{-2}}\times\dfrac{2-3\sqrt{-2}}{2-3\sqrt{-2}}=\dfrac{6-11\sqrt{-2}-6}{4+18}=\dfrac{11\sqrt{-2}}{22}=\dfrac{\sqrt{-2}}{2}=\dfrac{\sqrt{2}}{2}\sqrt{-1}$.

23. We have here merely to add the indices. 24. Multiply the index by n. 25. The expressions, or their conjugates, produce when multiplied together the same result as in Exercise 21.

XII. SIMPLE EQUATIONS, p. 143.

1. Numerical Equations.

26, 29. See the remarks on Ex. 6 and 4, p. 140. 33. Multiply by $4x$, and by $x+1$. 34. Multiply by 15; transpose; multiply by $7x-6$. 35. Multiply by 15, then $13(x-5)=9x+15$. 37. Multiply by the denominators, separately, simplifying. 47. Multiply by 30 and transpose, then $\dfrac{30x-15}{7}+6x=47$. 48. Multiply first by 3 and then by 12. 49. Multiply by 10, by 3, and by 23, incorporating at each step. 50. Multiply by 5, by $x+1$, and by 3; then $-5x-75x+\dfrac{240x^2+303x+63}{3x+2}=45$. 51. Transpose, double the terms of the right member, $\times$ by 2, then $x-\dfrac{3x^2+6}{3x-2}=\dfrac{-\frac{x}{3}-3}{x-1}$ or $-\dfrac{2x+6}{3x-2}=\dfrac{-x-9}{3(x-1)}$. 52. Multiply both terms of the first fraction by 3, of the second by 2, and of the third by 5; then multiplying by 11, we have $\dfrac{99x-22(x+1)}{12}+2-\frac{4}{5}x=12+\frac{1}{5}(x-1)$ which is easily solved. 53. Multiply by 12, transpose $4x$, multiply by 3, reduce the first fraction, incorporate; then $(17x-32)30=468x-792$. 54. Multiplying by $(x+2)(x+3)$, we get $4x+12+7x+14=37$. 55. Multiply

by 28, transpose, incorporate, then $154x + 294 = 49(3x + 7)$. 56. Multiply by 28, transpose, incorporate, and $98x - 42 = 14(3x + 1)$. 57. Subtracting, in each member, the second fraction from the first, we have $\frac{-1}{x^2 - 5x + 6} = \frac{-1}{x^2 - 13x + 42}$.

58. Transposing $\cdot15x - \cdot875x - \cdot0625x = -1 - \cdot2 - \cdot375$; or, $\cdot15x - \cdot9375x = -1\cdot575$, or $-\cdot7875x = -1\cdot575$. 59. Multiply by 7, by 9, transpose 14 and -72; multiply by $\cdot4$ and $81x \times \cdot4 + 63x + 12\cdot6 = 58 \times \cdot4$. 60. Multiply by $\cdot5$, then $\cdot6x - \cdot18x + \cdot05 = \cdot2x + 4\cdot45$; or, $\cdot6x - \cdot18x - \cdot2x = 4\cdot4$, that is $\cdot22x = 4\cdot4$ and $x = 20$.

2. Literal Equations.

. . . 73. $a - b = (a + b)x^{-2}$ $\therefore x^{-2} = \frac{1}{x^2} = \frac{a-b}{a+b}$ $\therefore x^2 = \frac{a+b}{a-b}$ and $x = \sqrt{\frac{a+b}{a-b}}$ . . . 79. Clearing and combining, $ax + bx = ac + \frac{a^2c}{b}$ $\therefore abx + b^2x = abc + a^2c$. Hence, separating the factors $b(a+b)x = ac(a+b)$, or $bx = ac$ $\therefore$ &c. . . . 83. Multiply by the L. C. M., $a(a+b)^3$; then $3a^2bc(a+b)^2 + a^3b^2 + (2a+b)b^2x(a+b) = 3acx(a+b)^3 + b(a+b)^3x$; transposing and separating the factors we get $x\{3ac(a+b)^3 + b(a+b)^3 - (2a+b)b^2(a+b)\} = 3a^2bc(a+b)^2 + a^3b^2$, separating the factor $a + b$ in the left member and a^2b in the second, this becomes $x(a+b)\{3ac(a+b)^2 + b(a+b)^2 - (2a+b)b^2\} = a^2b\{3c(a+b)^2 + ab\}$ now $b(a+b)^2 - (2a+b)b^2 = a^2b$; therefore $x(a+b)\{3ac(a+b)^2 + a^2b\} = a^2b\{3c(a+b)^2 + ab\}$. Now both sides of this equation are obviously divisible by a and by $3c(a+b)^2 + ab$; dividing we get $x(a+b) = ab$.

3. Equations with Surds.

. . . 91. Squaring, $x + a = x + 2a\sqrt{x} + a^2$;$\therefore \frac{1}{2}(1-a) = \sqrt{x}$ $\therefore$ &c. 93. Square, transpose 1, square again, &c. . . . 98. Squaring $a + x + 2\sqrt{(a^2 - x^2)} + a - x = b^2$. 99. Clearing and transposing $\sqrt{(7x + x^2)} = 1 - x$; $\therefore$ &c. 100. Clearing, &c., $\sqrt{(ax + x^2)} = x - \sqrt{a}$ $\therefore$ &c. . . . 102. Clearing we have $4ab + b\sqrt{x} + 4a\sqrt{x} + x = 6ab + 3b\sqrt{x} + 2a\sqrt{x} + x$ $\therefore$ $2a\sqrt{x} = 2ab + 2b\sqrt{x}$, $\therefore (a-b)\sqrt{x} = ab$. 103. Clear by mul-

tiplying $\sqrt{ax}-b$ by $\sqrt{ax}+5b$, and $\sqrt{ax}-2b$ by $\sqrt{ax}+3b$; then $ax + 4b\sqrt{ax}-5b^2 = ax + b\sqrt{ax}-6b^2 \therefore 3b\sqrt{ax} = -b^2 \therefore$ &c. 104. Employ the principle of Ex. 3, p. 139, then $\frac{2x^{\frac{1}{2}}}{2b^{\frac{1}{2}}} = \frac{x^{\frac{1}{2}}}{b^{\frac{1}{2}}} = \frac{a+b}{a-b} \therefore$ &c. 106. Here $\sqrt{(5x+10)} = \sqrt{5x}+2 \therefore 5x+10 = 5x + 4\sqrt{5x}+4 \therefore$ &c. 107. Similar to 99. 108. Raise to fourth power, transpose -1 and cube both sides then $\sqrt{7x}+2 = 8 \therefore$ &c. 109. Here $\sqrt{\frac{a^2-x^2}{a-x}} = \sqrt{(a+x)} = \sqrt{(x+a)} \therefore (x+a)+x-a = 2x \therefore \sqrt{(x+a)} = x+a \therefore 1 = \sqrt{(x+a)} \therefore$ &c. 110. Similar to 108; square, transpose, cube, transpose again, square again and transpose 20; then $\sqrt[4]{(3x+25)} = 5 \therefore$ &c. 111. Clearing we have $x + x^{\frac{1}{2}} - (x^2-x)^{\frac{1}{2}} = \frac{3x^{\frac{1}{2}}}{2} \therefore x - \frac{x^{\frac{1}{2}}}{2} = (x^2-x)^{\frac{1}{2}}$; divide by $x^{\frac{1}{2}}$, then $x^{\frac{1}{2}} - \frac{1}{2} = (x-1)^{\frac{1}{2}}$ square and $x - x^{\frac{1}{2}} + \frac{1}{4} = x-1 \therefore$ &c. 112. Here (Note, p. 32), $\sqrt{3x}+4$ is a factor of $3x-16$, the other factor being $\sqrt{3x}-4, \therefore \sqrt{3x}-4 = a + \frac{\sqrt{3x}-4}{a} \therefore a(\sqrt{3x}-4) = a^2 + \sqrt{3x} - 4 \therefore (a-1)(\sqrt{3x}-4) = a^2 \therefore$ &c. 113. Here both sides must be cubed; to do this most readily use the form in Ex. 4, p. 91, and Ex. 13, p. 142. We thus get $(a+x)^2 + (a-x)^2 + 3\sqrt[3]{(a^2-x^2)^2}\{\sqrt[3]{(a+x)^2} + \sqrt[3]{(a-x)^2}\} = 27(a^2-x^2)$. For the factor within the brackets use its equal, the right member of the given equation, and we have $(a+x)^2 + (a-x)^2 + 3\sqrt[3]{(a^2-x^2)^2} \cdot 3\sqrt[3]{(a^2-x^2)} = 27(a^2-x^2)$. Now the product of these two surds is the first power of the quantity; so that $(a+x)^2 + (a-x)^2 + 9(a^2-x^2) = 27a^2 - 27x^2$. 114. Similar to Ex. 14, p. 143.

115. Here as in Ex. 113, $1+x+1-x+3\sqrt[3]{(1-x^2)}\{\sqrt[3]{(1+x)} + \sqrt[3]{(1-x)}\} = 2$. Then, substituting $\sqrt[3]{2}$ for the part within the brackets, we have $2 + 3\sqrt[3]{(1-x^2)} \times \sqrt[3]{2} = 2 \therefore (27-27x^2)\cdot 2 = 0 \therefore x = 1$. 116. Use the artifice of Ex. 3, p. 139, and Ex. 11, p. 141, then $\frac{\sqrt[3]{(x+1)}}{\sqrt[3]{(x-1)}} = \frac{3}{1} \therefore$ &c. 117. Employing here the same artifice and since $b^2 = \frac{b^2}{1}$ we have $\frac{a+x}{\sqrt{(2ax+x^2)}} = \frac{b^2+1}{b^2-1}$ by omitting 2. Squaring and actually

dividing in left member, $\frac{a^2}{2ax+x^2}+1=\left(\frac{b^2+1}{b^2-1}\right)^2$. Transpose and reduce to com. denr., then $\frac{a^2}{2ax+x^2}=\frac{(b^2+1)^2}{(b^2-1)^2}-\frac{(b^2-1)^2}{(b^2-1)^2}=\frac{4b^2}{(b^2-1)^2}\ \therefore\ \frac{2ax+x^2}{a^2}=\frac{(b^2-1)^2}{4b^2}$. Add $\frac{a^2}{a^2}$ to the left member and its equal 1 to the right, then $\left(\frac{a+x}{a}\right)^2=\frac{(b^2-1)^2}{4b^2}+1=\frac{(b^2+1)^2}{4b^2}\ \therefore\ \frac{a+x}{a}=\frac{b^2+1}{2b}\ \therefore\ 2bx=a\,(b^2+1)-2ba\ \therefore\ 2bx=a(b^2-2b+1)=a(b-1)^2\ \therefore$ &c. 118. Transposing the first term of left member, and squaring $1-x+\sqrt{(1+x)}=1+x+1-x-2\sqrt{(1-x^2)}$; or $1+x-\sqrt{(1+x)}=2\sqrt{(1-x^2)}$; divide by $\sqrt{(1+x)}\ \therefore\ \sqrt{(1+x)}-1=2\sqrt{(1-x)}$. Squaring and incorporating, $-2\sqrt{(1+x)}=2-5x$, $\therefore$ &c.

119. Transpose the second term to the right side and square; then $(1+a)^2+x-ax=4a^2-4a\sqrt{(1-a)^2+(1+a)x}+(1-a)^2+x+ax\ \therefore\ 2\sqrt{(1-a)^2+(1+a)x}=x-2(1-a)\ \therefore\ 4(1-a)^2+4x+4ax=x^2-4x+4ax+4(1-a)^2\ \therefore$ &c.

120. Transpose; $x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}-\sqrt{(a^2-x^2)}$; square, $a^2x^2-x^2=a^4-a^4x^2+a^2-x^2-2a^2\sqrt{(1-x^2)(a^2-x^2)}$; transpose, $2a^2\sqrt{(1-x^2)(a^2-x^2)}=a^4+a^2-a^4x^2-a^2x^2=(1+a^2)a^2-(1+a^2)a^2x^2=(1+a^2)(a^2-a^2x^2)=(1+a^2)(1-x^2)a^2$. In this form both sides are divisible by a^2, and $\sqrt{(1-x^2)}$; hence $2\sqrt{(a^2-x^2)}=(1+a^2)\sqrt{(1-x^2)}$; squaring, $4(a^2-x^2)=(1+a^2)^2\cdot(1-x^2)=(1+a^2)^2\times1+(1+a^2)^2\times(-x^2)=(1+a^2)^2-(1+a^2)^2x^2$; transpose, $(1+a^2)^2x^2-4x^2=(1+a^2)^2-4a^2$; or, $\{(1+a^2)^2-4\}x^2=(1+a^2)^2-4a^2$; now, $(a^2+1)^2-4a^2=(a^2-1)^2$; and $\{(a^2+1)^2-4\}x^2=(a^2+3)(a^2-1)x^2\ \therefore\ (a^2+3)\cdot(a^2-1)x^2=(a^2-1)^2\ \therefore\ (a^2+3)x^2=a^2-1\ \therefore$ &c. 121. Squaring we have $\frac{b}{a+x}+\frac{c}{a-x}+2\sqrt{\frac{bc}{a^2-x^2}}=\sqrt{\frac{4bc}{a^2-x^2}}=2\sqrt{\frac{bc}{a^2-x^2}}$; hence $\frac{b}{a+x}+\frac{c}{a-x}=0\ \therefore$ &c. 122. Squaring, $\frac{x+1}{x-1}+\frac{x-1}{x+1}+2\sqrt{\frac{x^2-1}{x^2-1}}=a^2$, that is $\frac{2x^2+2}{x^2-1}+2=a^2\ \therefore\ (a^2-4)x^2=a^2$.

123. Square, reject x, divide by 4 and transpose, then $\sqrt{(bx+x^2)}=x-a+b$; $\therefore\ bx+x^2=x^2-2ax+2bx+a^2+b^2-2ab$, and $(2a-b)x=(a-b)^2$.

XIII. SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE, p. 159.

10. Multiply (1) by d and (2) by a to eliminate x and find y; then (1) by f and (2) by b to find x. 11. See note, p. 25. 19. Multiply (2) by 20 and add, to eliminate y. 21. Simple expressions are found by squaring each. 24. Here $(m^2 - n^2)x = (mb - na)xy$; or, $m^2 - n^2 = (mb - na)y$, &c.; or thus, $\frac{m^2 - n^2}{x} = ma - nb$ without first clearing. 29. Transpose x and square in both; y is then easily eliminated and x found. 31. Clear (1) and equate the values of ax. 32. Multiply the terms of $b(x+y) = a(x-y)$ by $x+y$; then $b(x+y)^2 = a(x^2 - y^2) = ac$ from (2) $\therefore (x+y)^2 = \frac{ac}{b} = \frac{a^2c}{ab}$; hence $x + y = a\sqrt{\frac{c}{ab}}$ (3). Again, multiply by $x - y$ and $b(x^2 - y^2) = a(x-y)^2$; or $bc = a(x-y)^2$; $\therefore (x-y)^2 = \frac{bc}{a} = \frac{b^2c}{ab}$ and $x - y = b\sqrt{\frac{c}{ab}}$ (4), then combine (3) with (4). 33. Clearing and transposing $(ab + bc)x + acy = abc \ldots (3)$; $acx - (ab - bc)y = abc \ldots (4)$; multiply (3) by $ab - bc$ and (4) by ac; then $x(a^2b^2 - b^2c^2) + ac(ab - bc) \times y = (ab - bc)abc \ldots (5)$; $a^2c^2x - ac(ab - bc)y = ac \cdot abc \ldots (6)$. Add (5) and (6), collect the coeffs. of x and divide. To find y, multiply (3) by ac and (4) by $ab + bc$, and proceed as in finding x. 34. The first reduced becomes $11x - 7y = 0$. 35. The reduced equations are $4x + y = 17$; and $2x - 3y = -9$. 36. The equations reduced are $21y - 13x = 3$ and $9y + 4x = 30$. 37. From (1) $(a^2 - b^2)5x + (a^2 - b^2)3y = 8a^2b - 2ab^2 \ldots (3)$; multiply (2) by 3 and transpose, then $(a^2 - b^2)3y + 3(a + b + c)bx = 3a^2b + 6ab^2 + \frac{3ab^2c}{a+b} \ldots (4)$; subtract (4) from (3), $(5a^2 - 8b^2 - 3ab - 3bc)x = 5a^2b - 8ab^2 - \frac{3ab^2c}{a+b}$; or, $(5a^2 - 8b^2 - 3ab - 3bc)x = ab\left(5a - 8b - \frac{3bc}{a+b}\right) = \frac{ab}{a+b}(5a^2 - 8b^2 - 3ab - 3bc) \therefore x = \frac{ab}{a+b}$. Substitute this value in (3), then $5ab(a - b) + (a^2 - b^2) \times 3y = 8a^2b - 2ab^2, \therefore (a^2 - b^2)3y = 3a^2b + 3ab^2 = 3ab(a + b)$. 38. Here $3\frac{1}{3} : 3\frac{1}{2} :: 20 : 21$. The cleared equations are $10x - 18y = -2$, $308x - 31y = 2032$.

39. Substitute in (2) the value of x in (1) then $10{\cdot}4888 - {\cdot}56y + 13{\cdot}8121y = 763{\cdot}4$ $\therefore$ $13{\cdot}2521y = 752{\cdot}9112$. 40. Substitute in (2) the value of x in (1), and multiply the result by bc $\therefore$ $c(b^3 - c^3)y = 2b^4 - 2bc^3 - ac^3 + ab^3 = (a + 2b)(b^3 - c^3)$ $\therefore$ &c. 41. Multiply the members of (1) by 3, transpose; multiply again by $9 - 2y$; then $218y - 13x = 999$. Again multiply the members of (2) by 18, transpose, multiply again by $4x - 10$ and $72y + 4x = 388$. 42. The reduced equations are $1011x - 638y = 1335$ and $9x - 10y = -27$.

43. Multiply (1) by 20; $\frac{35x + 120y}{5} - \frac{12y + 24 - 6x + 4}{8} = 100 - \frac{5x}{4}$ clear, transpose, divide by 180; and $2x + 5y = 23$. In (2) the last ratio is that of 63 to 7 or $9 : 1$; hence, $\frac{3x}{2} + \frac{2y}{3} + \frac{5}{2} = \frac{9x}{2} - \frac{9y}{3} + \frac{3}{2}$. 44. Multiply (1) by $11x$, then $44x + 12xy - 6y - 2 = 11xy + \frac{3xy - 31 + 110x + 143}{3}$; whence $22x - 18y = 118$. Multiply now (2) by 3 and by $6y + 27$, then $12xy + 54x - \frac{54xy - 90y + 243x - 405}{y + 7} = 12xy + 170$; whence $135x - 80y = 785$. 45. For Eq. (1) multiply by 3 and by 7; then multiply by 20 and $620x - 580y = 105x - 84y - 2634$, or, $515x - 496y = -2634$. For Eq. (2) take the sum and difference of the terms, as in Exs. 3, 11, pp. 139, 141, since $a : b$ is the same as $\frac{a}{b}$; this gives $30 : 4x - 2y :: \frac{5}{6} : \frac{2x}{3} - \frac{y}{2} + \frac{2}{3}$; or, $100x - 80y = -120$. 46. In (2) take the products, transpose and incorporate; then, $2\sqrt{(y - x)} = 3\sqrt{(20 - x)}$...(3), squaring, &c., $4y + 5x = 180$...(4). Next double the terms of (1); $2\sqrt{y} - 2\sqrt{(20 - x)} = 2\sqrt{(y - x)}$; but (3) $2\sqrt{(y - x)} = 3\sqrt{(20 - x)}$, $\therefore$ $2\sqrt{y} - 2\sqrt{(20 - x)} = 3\sqrt{(20 - x)}$, $\therefore$ $2\sqrt{y} = 5\sqrt{(20 - x)}$. Whence, $4y + 25x = 500$....(5); combine (4) and (5). 51. Multiply (1) by 3, and take (2) from the result; then multiply (1) by 4, and take (3) from the result; this gives $2x + y = 52$, and $3x + y = 75$. 55. From (2) take (1), and combine the result with (3) to find z. 60. Multiply (2) by 3, and with (3) eliminate z; then multiply (2) by 4, and with (1) eliminate z. 61. In (2) for $2x$ put $3z$. . . .

63. Here $\frac{1}{y} - \frac{1}{z} = a - b$; combine this with (3) and $\frac{2}{y} = a - b + c$, &c. 66. Here $y = \frac{nx}{m}$, $z = \frac{qx}{p}$; substitute these in (3) to find x; for y and z multiply the value of x by $\frac{n}{m}$ and $\frac{q}{p}$. 67. The eliminating equations are, for x, $adx + bdy = dc$, and $adx + aey = af$, giving y; for y, $aex + bey = ce$, and $bdx + bey = bf$, giving x, hence $z = \frac{l}{h} - \frac{g}{h} \cdot \frac{af - dc}{ae - bd} = \frac{lae - lbd - gaf + gdc}{h(ae - bd)}$ $\therefore$ &c. 68. See Alg., p. 163, note; and compare Ex. 18, p. 50. 69. To eliminate z, multiply (2) by 3, and add the result to (1); then multiply (2) by 5, and add the result to (3); next multiply (1) by 2, and (4) by 3, and add; the three resulting equations are, $7x + 12y - 13u = 2070$, $13x + 36y - 3u = 3610$, $14x - 36y - 47u = 1590$; hence, easily, $27x - 50u = 5200$, $35x - 86u = 7800$ $\therefore$ &c. 71. Take the reciprocals of the given equations; then, $\frac{x + y}{xy} = 1$; that is, $\frac{x}{xy} + \frac{y}{xy} = 1$ $\therefore$ $\frac{1}{y} + \frac{1}{x} = 1$. Similarly, $\frac{1}{z} + \frac{1}{x} = \frac{1}{2}$, $\frac{1}{z} + \frac{1}{y} = \frac{1}{3}$ $\therefore \frac{1}{y} - \frac{1}{z} = \frac{1}{2}$ $\therefore \frac{2}{y} = \frac{5}{6}$ &c. 72. Multiply (1) by 11, (2) by (-2), and (3) by (-133) and add the results; then, $-86x = -86$, and $x = 1$. From (3), $-\sqrt{y} + \sqrt[3]{z} = 1$ $\therefore$ $-13\sqrt{y} + 13\sqrt[3]{z} = 13$...(4), from (1), $-11\sqrt{y} + 13\sqrt[3]{z} = 17$...(5); take (4) from (5), $2\sqrt{y} = 4$ $\therefore$ &c. 73. Multiply (2) by 3, and add the product to (4), then $24y - 15z - 4u = 43$...(6); to $3 \times (3)$ add (1) and double the result, then $24y - 4z + 14u = 112$...(7), from (7) take (6), then $11z + 18u = 69$...(8), with this take (5), $7z - 5u = 11$. Whence z easily, and thence the others. 74. Divide both members of the equations, severally, by axy, bxz, cyz, and we have the three following equations, $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$; $\frac{1}{z} + \frac{1}{x} = \frac{1}{b}$; $\frac{1}{y} + \frac{1}{z} = \frac{1}{c}$; whence the values are easily found as in Ex. 63 and 71. 75. Here $(x + y + z)(x + y + z) = a^2 + b^2 + c^2$ $\therefore$ $x + y + z = \sqrt{(a^2 + b^2 + c^2)}$ $\therefore$ from (1) $x\sqrt{(a^2 + b^2 + c^2)} = a^2$; whence x, and so of the others. 76. Here $xyz = 3 \cdot 7 \cdot 11$; $xyw = 2^2 \cdot 3 \cdot 5 \cdot 7$ $yzw = 2^2 \cdot 5 \cdot 7 \cdot 11$; $xzw = 2^2 \cdot 3 \cdot 5 \cdot 11$; $\therefore$ $w^3x^3y^3z^3 = 2^6 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11^3$; $\therefore$ $wxyz = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$; $\therefore$ $\frac{wxyz}{xyz} = w = 2^2 \cdot 5 = 20$, &c. 77. 1. The equations are not independ-

ent, the second being double the first. 2. They are incongruous; for $3x+3y$ must be 21. 3. Not independent; (2) is derived from (1) by adding a third part. 4. The conditions are inconsistent (1) and (3) would give $y=3$, and $\therefore x=4$; values which will not satisfy (2): if for 1 we put 9 in (2) the values would satisfy (2); but in that case one of the equations would be superfluous. 5. Here (3) does not give a separate condition; for it is the difference between (2) and 4 times (1). 6. Here from (1) and (2) we get $19x+19y=133$; from (1) and (3) $19x+19y=140$; from (2) and (3) $19x+19y=84$, inconsistent results; in fact for 12 we ought to have 5, the left member of (3) being the difference of those of the other two. If we had 5, then, in all the above cases we should have $19x+19y=133$, or $x+y=7$, the values being indeterminate. 7. The equations are not independent, the third being the sum of the second and twice the first.

XIV. QUADRATIC EQUATIONS, p. 175.

Exs. 1 to 6 are pure quadratics. Ex. 7 to 24 illustrate the first or common rule; Ex. 25 to 35 the second or "Hindoo" rule. In solving the others, the student will be guided by the suggestions in Art. 146. The method of substitution, Note, p. 166, and Note, p. 169, is preferred by many, and is often the most convenient, especially when the numbers are large in numerical equations. In such cases it is unnecessary to set down the middle term in completing the square. Exs. 48, 49, 50, 51; the reduced equations are $10x^2+57x=261$; $12x^3-73x=-100$; $x^2-16x=537$; $4x^2-11x=-7$. 52. The root is $\sqrt{x}+1=\pm 5$. Ex. 53. The root is $4x^2-7=\pm 29$. 54. Divide by $\sqrt{x}$; then $x^2+x=6$. 55. Clearing $3x+2\sqrt{x}=16$. 56. Divide by $\sqrt{x}$. 57. Multiply by $x^{\frac{3}{2}}$; then $\frac{8}{x^{\frac{3}{2}}}+2x^{\frac{3}{2}}=17$; multiply again by $x^{\frac{3}{2}}$, then $8+2x^3=17x^{\frac{3}{2}}$, $\therefore$ $2x^3-17x^{\frac{3}{2}}=-8$; $\therefore$ $16x^3-136x^{\frac{3}{2}}+289=225$, $\therefore$ $4x^{\frac{3}{2}}-17=\pm 15$, $\therefore$ $x^{\frac{3}{2}}=8$, or $\frac{1}{2}$, $\therefore$ $\sqrt{x^3}=8$, or $\frac{1}{2}$, $\therefore$ $\sqrt{x}=2$, or $\sqrt[3]{\frac{1}{2}}$, $\therefore$ $x=4$; or $(\sqrt[3]{\frac{1}{2}})^2$, *i.e.*, $(\frac{1}{2})^{\frac{2}{3}}=\sqrt[3]{\frac{1}{4}}$. 58. The root is $2x^{\frac{1}{5}}+1=\pm 5$. 59. The root is $2(x-5)^{\frac{3}{2}}+3=\pm 13$, $\therefore x-5=$

$8^{\frac{2}{3}}$, or $(-5)^{\frac{2}{3}}$, $\therefore x-5=4$, or $(-5)^{\frac{2}{3}}$ &c. See Ex. 57. 60. The root is $2x^3+3=\pm 57$. . . . 63. Divide by x^2; then $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0$; or $x^2+\frac{1}{x^2}+x+\frac{1}{x}-4=0$; now $x^2+\frac{1}{x^2}+x+\frac{1}{x}-4=x^2+2+\frac{1}{x^2}+x+\frac{1}{x}-6 \therefore \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=6$. Solving this we have $x+\frac{1}{x}=2$, or -3. The first of these gives $x^2-2x+1=0$; the second $x^2+3x=-1$ &c. 65. After clearing and completing the square we have $256x^2-\ldots\ldots+100a^2=4a^2 \therefore 16x-10a=\pm 2a$.

66. Transposing, $x^2-\frac{4}{9}=1+\frac{2}{3x}$; separating into factors, $(x+\frac{2}{3})(x-\frac{2}{3})=\frac{1}{x}(x+\frac{2}{3})$, divide by $x+\frac{2}{3}$; then $x-\frac{2}{3}=\frac{1}{x}$, $\therefore x^2-\frac{2}{3}x=1$. Also since the equation may have the form $(x+\frac{2}{3})(x-\frac{2}{3})-\frac{1}{x}(x+\frac{2}{3})=0$, it will be true if $x+\frac{2}{3}=0$, which gives $x=-\frac{2}{3}$. Again, it may be thus solved; clear and transpose, then $9x^3-13x-6=0$, $\therefore (3x+2)(3x^2-2x-3)=0$; put $3x+2=0$, $\therefore x=-\frac{2}{3}$; put $3x^2-2x-3=0$, $\therefore x^2-\frac{2}{3}x=1$ as above. 68. Multiply by $x-5$ and by 20, then $x-12\sqrt{x}=-35$. 69. Square both sides.

70. Divide by $x^{\frac{1}{3}}$; then $3x^{\frac{5}{3}}-x^{\frac{5}{6}}=3104$; $\therefore x^{\frac{5}{6}}=-32$, or $\frac{97}{3}$; $\therefore x=-32^{\frac{6}{5}}=(-2^5)^{\frac{6}{5}}=-2^6=64$, also $x=(\frac{97}{3})^{\frac{6}{5}}$. See Ex. 57, 59. 71. Here $x^3-3x^2-2=0$; $\therefore (x+1)(x^2-x-2)=0$. Now $x+1=0$ gives $x=-1$; and $x^2-x-2=0$ gives $x^2-x=2 \therefore$ &c. Or thus; add $2x$ to both sides, then $x^3-x=2x+2$; $\therefore x(x^2-1)=2(x+1)$; or $x(x+1)(x-1)-2(x+1)=0$. Now this is true if $x+1=0$, giving $x=-1$; but by division, $x(x-1)=2$, or $x^2-x=2$ &c. 72. Here $x^n+\frac{1}{x^n}=a \therefore x^{2n}-ax^n=1$ &c. 73. Apply Rule II. after clearing. 74. Here $x^3-1=(x-1)(x^2+x+1)=0 \therefore x-1=0$; and $x^2+x+1=0$, in all 3 values. 75. Here $ax^{\frac{5}{8}}-bx^{\frac{5}{4}}=cx^{\frac{15}{8}}$; divide by $x^{\frac{5}{8}}$; $\therefore a-bx^{\frac{5}{8}}=cx^{\frac{5}{4}}$, $\therefore cx^{\frac{5}{4}}+bx^{\frac{5}{8}}=a$, a common quadratic. See Ex. 57, 59, 70.

76. Divide by x^2, add 2 to both sides, then $x^2+2+\frac{1}{x^2}=2$. $\therefore$ $x+\frac{1}{x}=\pm\sqrt{2}$; $\times$ by x, then $x^2+1=\pm x\sqrt{2}$; and $x^2\mp\sqrt{2}.x=1$. Or thus add $2x^2$ to both sides and take the square root, then $x^2+1=\pm x\sqrt{2}$; $\therefore x^2\mp\sqrt{2}.x=-1$. Complete the square by Rule I.; then $x^2\mp\sqrt{2}.x+\left(\frac{\sqrt{2}}{2}\right)^2=-1+\left(\frac{\sqrt{2}}{2}\right)^2$; or $x^2\mp\sqrt{2}.x+\frac{1}{2}=-\frac{1}{2}$ and $x\mp\sqrt{\frac{1}{2}}=\sqrt{(-\frac{1}{2})}=\sqrt{\frac{1}{2}}\sqrt{(-1)}$; but $\sqrt{\frac{1}{2}}=\frac{\sqrt{1}}{\sqrt{2}}=\frac{1}{\sqrt{2}}$, $\therefore x\mp\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}}\sqrt{(-1)}$, $\therefore$ &c. 77. If $\frac{x^2}{4}$ be added to both sides, the left will become the square of $x^2-\frac{3x}{2}-1$, and we have $x^2-\frac{3}{2}x-1=\pm\frac{5}{6}x$ which gives the two equations $3x^2-7x=3$, and $3x^2-2x=3$. Otherwise, add $\frac{9x^2}{4}$ to the left member and also subtract it; then $\left(x^2-\frac{3x}{2}\right)^2-\frac{9x^2}{4}+3x+1=\frac{4x^2}{9}$; now $-\frac{9x^2}{4}=-2x^2-\frac{x^2}{4}$; hence $\left(x^2-\frac{3x}{2}\right)^2-2\left(x^2-\frac{3x}{2}\right)-\frac{x^2}{4}+1=\frac{4x^2}{9}$; $\therefore (x^2-\frac{3}{2}x)^2-2(x^2-\frac{3}{2}x)+1=\frac{25}{36}x^2$, $\therefore$ $x^2-\frac{3x}{2}-1=\pm\frac{5}{6}x$; the same equation as before. 78. Square and divide by a^2x^4, then $\frac{4}{a^2x^2}-\frac{4x^2}{a^2}=\frac{1}{x^4}+2+x^4$ or $-\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right)=\left(x^2+\frac{1}{x^2}\right)^2$, or $\left(x^2+\frac{1}{x^2}\right)^2+\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right)=0$; but $\left(x^2+\frac{1}{x^2}\right)^2=\left(x^2-\frac{1}{x^2}\right)^2+4$, $\therefore$ $\left(x^2-\frac{1}{x^2}\right)^2+\frac{4}{a^2}\times\left(x^2-\frac{1}{x^2}\right)=-4$; complete the square, $\left(x^2-\frac{1}{x^2}\right)^2+\frac{4}{a^2}\left(x^2-\frac{1}{x^2}\right)+\left(\frac{2}{a^2}\right)^2=\frac{4}{a^4}-4=\frac{4}{a^4}-\frac{4a^4}{a^4}=\frac{4}{a^4}(1-a^4)$; taking the root, $\left(x^2-\frac{1}{x^2}\right)+\frac{2}{a^2}=\pm\frac{2}{a^2}\sqrt{(1-a^4)}$, $\therefore x^2-\frac{1}{x^2}=-\frac{2}{a^2}\{1\mp\sqrt{(1-a^4)}\}$; put this $=2b^2$, then $x^2-\frac{1}{x^2}=2b^2$; clear and transpose; x^4-

$2b^2x^2 = 1$; add b^4 to both sides; then $x^4 - 2b^2x^2 + b^4 = 1 + b^4, \therefore x^2 - b^2 = \pm\sqrt{(1+b^4)} \therefore x = \pm\sqrt{(b^2 \pm \sqrt{1+b^4})}$. Now, $b^2 = -\frac{1}{a^2}\{1 \mp \sqrt{(1-a^4)}\} \therefore 1 + b^4 = 1 + \frac{1}{a^4}(2 - a^4 \mp 2\sqrt{1-a^4}) = \frac{2}{a^4} \times (1 \mp \sqrt{1-a^4})$, since $\frac{1}{a^4} \times (-a^4) = -1$. Putting in the expression for x, these values of b^2 and $1+b^4$, and since $\sqrt{a^2} = a$, we have the value required. Otherwise thus, square and multiply by a^2, then $a^4(1+x^4)^2 = 4a^2x^2(1-x^4)$; or $a^4(1+2x^4+x^8) = 4a^2x^2(1-x^4)$, or $a^4 + 2a^4x^4 + a^4x^8 = 4a^2x^2(1-x^4)$. From both sides take $4a^4x^4$; then $a^4 - 2a^4x^4 + a^4x^8 = 4a^2x^2(1-x^4) - 4a^4x^4$; that is, $a^4(1-x^4)^2 - 4a^2x^2(1-x^4) = -4a^4x^4$. Add now $4x^4$ to both sides, and take the square root of each member; then $a^2(1-x^4) - 2x^2 = \pm 2x^2\sqrt{(1-a^4)}$; transpose all the terms to one side, multiply by a^2, and transpose a^4 to the right side, then $a^4x^4 + 2a^2x^2(1 \pm \sqrt{1-a^4}) = a^4$; complete the square, $a^4x^4 + 2a^2x^2(1 \pm \sqrt{1-a^4}) + (1 \pm \sqrt{1-a^4})^2 = a^4 + (1 \pm \sqrt{1-a^4})^2 = a^4 + 1 + 1 - a^4 \pm 2\sqrt{1-a^4} = 2 + 2\sqrt{1-a^4} = 2(1 \pm \sqrt{1-a^4})$; taking the root, $a^2x^2 + 1 \pm \sqrt{1-a^4} = \pm\sqrt{\{2(1 \pm \sqrt{1-a^4})\}}$; hence $a^2x^2 = -1 \mp \sqrt{1-a^4} \pm \sqrt{\{2(1 \pm \sqrt{1-a^4})\}}$ $\therefore$ &c. 79. The reduced equation is $7x^2 - 39x = 70$.

80. The reduced equation is $9x^2 - 7x = 116$. 81. Multiply by 3, 5, 38, and $10 - x$; so reduced the equation is $x^2 - 19x = 290$. 82. This equation may be resolved in two ways; *first*, $x^6 - 1 = (x^3 - 1)(x^3 + 1) = 0 \therefore x^3 - 1 = 0$. See Ex. 74. Also $x^3 + 1 = 0$, or $(x+1)(x^2 - x + 1) = 0$; hence $x + 1 = 0$, and $x^2 - x + 1 = 0$, which will give the roots. *Secondly*, the equation may be resolved into the three quadratic factors $x^2 + 1$, $x^2 + x + 1$, and $x^2 - x + 1$, the solution of which will give the six roots. 84. Here $x^2 + 7ax = 60a^2$. 85. Here $x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = 6\frac{1}{4} = \frac{25}{4}, \therefore x^{-\frac{1}{2}} + \frac{1}{2} = \pm\frac{5}{2}; \therefore x^{-\frac{1}{2}} = 2$ or -3 and $x^{\frac{1}{2}} = \frac{1}{2}$ or $-\frac{1}{3}$, &c. 86. Square, transpose, square; $4x^2 - 4ax = 2ab - a^2 - b^2$; and a^2 will complete the square. 87. Here $x^2 - 8ax = 3b^2$. 89. Cube, transpose, divide by $3b$, $x^2 - bx = \frac{a^3 - b^3}{3b}, \therefore$ &c. 90. Multiply by abx and by $a + b + x$, arrange the terms, cancel abx; then $0 = (a^2 + 2ab + b)x + ab(a+b) + (a+b)x^2$; divide by $a + b$ then $x^2 + (a+b)x = -ab$. 91. Here $4m^2q^2x^2 \ldots + (mn - pq)^2 = 4mnpq + (mn - pq)^2 = (mn + pq)^2 \therefore 2mqx$

$-(mn-pq) = \pm(mn+pq)$ $\therefore$ &c. 92. Multiply, transpose, simplify; $(a^2-b^2)x^2 - 2(a^2+b^2)x = b^2 - a^2 = -(a^2-b^2)$ $\therefore$ 4 $(a^2-b^2)^2x^2 - 8(a^4-b^4)x + 4(a^2+b^2)^2 = -4(a^2-b^2)^2 + 4(a^2+b^2)^2$ $\therefore$ &c. 93. Clearing by 7, 3, and 5, we have $5x^2+2x=3$. 94. Multiply by $3-x$ and by $19-7x$; then $\frac{342-107x-7x^2}{6} + \frac{1235-455x}{4} = 51x - 20x^2 + 27$, multiply by 12, transpose and incorporate them $226x^2 - 2191x = -4065$. 95. The products give $x^2 - \frac{5}{6}x + \frac{1}{6} + x^2 - \frac{7}{12} + \frac{1}{12} = x^2 - \frac{9}{20}x + \frac{1}{20}$, the reduced equation is $30x^2 - 29x = -6$. 96. See Exs. 3, 6, p. 171-2. 97. Add 5 to both sides, see Ex. 6, p. 172. 98. Multiply the rational part successively by 3, $8a$, and the reduced part by 9 and $64a^2$; then $3x^2 + 16ax + 12a^2 = \sqrt{(192 \times ax^3 + 144a^2x^2)}$; square both sides, transpose, and incorporate, and $9x^4 - 96ax^3 + 184a^2x^2 + 384a^3x + 144a^4 = 0$, *i.e.*, $(3x^2 - 16 \times ax - 12a^2)^2 = 0$. $\therefore 3x^2 - 16ax = 12a^2$; otherwise, resolve the radical part into the factors $\frac{x}{\sqrt{2a}}\sqrt{\frac{2x}{3}+\frac{a}{2}}$, and transpose, then $\frac{2x}{3}+\frac{a}{2} - \frac{x}{\sqrt{2a}}\sqrt{\frac{2x}{3}+\frac{a}{2}} + \frac{x^2}{8a} = 0$; now $\left(-\frac{x}{2\sqrt{2a}}\right) \times \left(-\frac{x}{2\sqrt{2a}}\right) = \frac{x^2}{8a}$; $\therefore \sqrt{\frac{2x}{3}+\frac{a}{2}} - \frac{x}{2\sqrt{2a}} = 0$, and $\sqrt{\frac{2x}{3}+\frac{a}{2}} = \frac{x}{2\sqrt{2a}}$ and $\frac{2x}{3}+\frac{a}{2} = \frac{x^2}{8a}$; whence $3x^2 - 16ax = 12a^2$, as before. 99. Multiply by $\sqrt{(x^2-1)}$ $\therefore$ $\sqrt{(x^2-1)} + \sqrt{(x-1)} = \sqrt{x^3}$; square, $x^2 - 1 + x - 1 - 2\sqrt{(x^3-x^2-x+1)} = x^3$, $\therefore x^3 - x^2 - x + 1 - 2\sqrt{(x^3-x^2-x+1)} + 1 = 0$, take the root, $\sqrt{(x^3-x^2-x+1)} - 1 = 0$, $\therefore x^3 - x^2 - x + 1 = 1$, $\therefore x^2 - x = 1$, $\therefore$ &c. 100. Here Rule II., $4\overline{(x-2)^2-x}^2 - 4\overline{(x-2)^2-x} + 1 = 361$, $\therefore 2\overline{(x-2)^2-x} - 1 = \pm 19$; $\therefore (x-2)^2 - x = 10$ or -9; hence $x^2 - 5x = 6$, and $x^2 - 5x = -13$. 101. Transpose, $a + x - \sqrt{(2a^2-ax-x^2)} = \sqrt{(ax-x^2)} - \sqrt{(2ax+x^2)}$; square; $a^2 + 2ax + x^2 + 2a^2 - ax - x^2 - 2(a+x)\sqrt{(2a^2-ax-x^2)} = ax - x^2 + 2ax + x^2 - 2\sqrt{(2a^2x^2 - ax^3 - x^4)}$ $\therefore$ by incorporating $3a^2 + ax - 2(a+x)\sqrt{(2a^2-ax-x^2)} = 3ax - 2\sqrt{(2a^2x^2-ax^3-x^4)}$ $\therefore 3a^2 - 2ax = 2(a+x)\sqrt{(2a^2-ax-x^2)} - 2\sqrt{\{x^2(2a^2-ax-x^2)\}}$, separating the factor x^2 within the second vinculum, but $2\sqrt{x^2} = 2x$, and $2(a+x) - 2x = 2a + 2x - 2x$ $\therefore 3a^2 - 2ax = (2a + 2x - 2x)\sqrt{(2a^2-ax-x^2)}$ or $3a^2 - 2ax = 2a\sqrt{(2a^2-ax-x^2)}$. Divide by a, square and trans-

pose; then, $8x^2 - 8ax = -a^2$. 102. Developing by continual multiplication or by the binomial theorem, Art. 92, and incorporating like terms, we have $2x^5 + 20x^3 + 10x = 19(2x^3 + 6x)$; or dividing by $2x$, and transposing, $x^4 - 9x^2 = 52$. 103. Multiply by the first denominator, then $(27a + 8x)^{\frac{2}{3}} + \dfrac{40x}{\sqrt[3]{(27a+8x)}} = 24x^{\frac{2}{3}}$; clear, $27a + 8x + 40x = 24x^{\frac{2}{3}}(27a + 8x)^{\frac{1}{3}}$, add, divide by 3, $9a + 16x = 8x^{\frac{2}{3}}(27a + 8x)^{\frac{1}{3}}$; put now b for $9a$ and y for $8x$ and cube; then $b^3 + 6b^2y + 12by^2 + 8y^3 = 24by^2 + 8y^3, \therefore y^2 - \frac{b}{2}y = \frac{b^2}{12}$; this quadratic gives $y = \frac{b}{4} \pm \sqrt{\frac{7b^2}{48}} = \frac{b}{4}(1 \pm \sqrt{\frac{7}{3}}) = \frac{b}{4} \times (1 \pm \frac{1}{3}\sqrt{21}) = \frac{b}{4} \pm \frac{1}{12}b\sqrt{21} = \frac{b}{12}(3 + \sqrt{21}) = \frac{9a}{12}(3 \pm \sqrt{21}) = \frac{3a}{4} \times (3 \pm \sqrt{21}), \therefore x = \frac{y}{8} = \frac{3a}{32}(3 \pm \sqrt{21})$. 104. Taking all the terms to the left side and taking the difference of the fractions, we have $\dfrac{10}{x^2-1} - \dfrac{16}{x^2-4} + \dfrac{126}{x^2-9} = 0$; this cleared is, $5(x^4 - 13x^2 + 36) - 8(x^4 - 10x^2 + 9) + 63(x^4 - 5x^2 + 4) = 0$, which reduces to $x^4 - 5x^2 + 6 = 0$. 105. Multiply by 2, transpose $3x$ and add 6 to both sides; then $2x^2 - 5x + 6 + 10\sqrt{(2x^2 - 5x + 6)} = 39$, by Rule I. $\sqrt{(2x^2 - 5x + 6)} + 5 = \pm 8$. This gives the two equations $2x^2 - 5x + 6 = 9$; and $2x^2 - 5x + 6 = 169$, and four values of the unknown. 106. Transpose first term to right side, square both sides, cancel $\frac{1}{x}$, then $1 = x^2 + x - 2x\sqrt{(x - \frac{1}{x})}$, divide by x and transpose, $x - \frac{1}{x} - 2\sqrt{(x - \frac{1}{x})} + 1 = 0$, a quadratic whose root is $\sqrt{(x - \frac{1}{x})} = 1$, $\therefore x^2 - x = 1$, &c. 107. Here $(1 + x)^{\frac{2}{n}} - (1 - x)^{\frac{2}{n}} = (1 - x^2)^{\frac{1}{n}} = (1 + x)^{\frac{1}{n}}.(1 - x)^{\frac{1}{n}}$; divide by $(1 - x)^{\frac{2}{n}}$; then $\left(\frac{1+x}{1-x}\right)^{\frac{2}{n}} - 1 = (1 + x)^{\frac{1}{n}}(1 - x)^{-\frac{1}{n}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{n}}$; transpose, $\left(\frac{1+x}{1-x}\right)^{\frac{2}{n}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{n}} = 1; \therefore$ Rule I., $\left(\frac{1+x}{1-x}\right)^{\frac{1}{n}} - \frac{1}{2} = \pm\sqrt{\frac{5}{4}} = \pm\frac{1}{2}\sqrt{5}; \therefore \left(\frac{1+x}{1-x}\right) = \frac{(1 \pm \sqrt{5})^n}{2^n}$. The answer is found by dividing the difference of the terms by their sum, Ex. 3, p. 139. 108. Clearing, $x + \sqrt{(2 - x^2)} - x + \sqrt{(2 - x^2)} = x^2 - (2 - x^2)$, that is, $2\sqrt{(2 - x^2)} = 2(x^2 - 1)$, or $\sqrt{(2 - x^2)} = x^2 - 1$; but $x^2 - 1 = x^2 - 2 + 1 = -(2 - x^2) + 1; \therefore \sqrt{(2 - x^2)} = -(2 - x^2) + 1$; transpose, $2 - x^2 + \sqrt{(2 - x^2)} = 1; \therefore$ Rule I.,

$\sqrt{(2-x^2)} = \frac{-1 \pm \sqrt{5}}{2}$; $\therefore 2 - x^2 = \frac{6 \pm 2\sqrt{5}}{4} = \frac{3 \pm \sqrt{5}}{2}$, &c. 109. Since $x^3 - 3x + 2 = (x^2 - 2x + 1)(x + 2)$, if 1 be transposed the equation will take the form, $x - 1 - 2\sqrt{(x+2)} = \sqrt[4]{(x^2 - 2x + 1)}$ $\sqrt[4]{(x+2)} = \sqrt[4]{(x-1)^2}. \sqrt[4]{(x+2)} = \sqrt{(x-1)}. \sqrt[4]{(x+2)}$. Squaring both sides in this latter form we have $(x-1)^2 - 4(x-1)$. $\sqrt{(x+2)} + 4(x+2) = (x-1).\sqrt{(x+2)}$; transpose, $4(x+2) - 5(x-1).\sqrt{(x+2)} = -(x-1)^2$, $\therefore$ Rule II., $64(x+2) - \ldots + 25(x-1)^2 = 9(x-1)^2$, $\therefore 8\sqrt{(x+2)} = \pm 3(x-1) + 5(x-1)$, $\therefore$ $\sqrt{(x+2)} = x - 1$ or $\frac{1}{4}(x-1)$; hence $x^2 - 2x + 1 = x + 2$, (a); and $x^2 - 2x + 1 = 16x + 32$, (b). These give $x^2 - 3x = 1$, (a^1); and $x^2 - 18x = 31$ . . . (b^1), which readily give the four values of x. It might seem at first sight that from the equations (a) and (b) above, we should have $16x + 32 = x + 2$; and $\therefore$ $x = -2$. But it must be remembered that x does not stand for the same quantity in both, as in the case of simultaneous equations, and that, therefore, the right members of (a) and (b) are not equal, indeed, such a value of x would be contradictory and give absurd results both here and in the given equations.

It ought to have been remarked in the Algebra in regard to surd equations that the values of x duly derived will not always satisfy the given equations; and in such a case there are no others that will satisfy it; the equation being incapable of solution. The equation $2x + \sqrt{(x^2 - 7)} = 5$, given in a note in *Hutton's Course of Mathematics*, by T. S. Davies, vol. i. p. 195, is an equation of this kind. See also Mr. Davies' supplementary vol., p. 238. Duly solved it gives the values 4 and $\frac{8}{3}$, but these will not satisfy the equation, as they give 11 and $\frac{17}{3}$ instead of 5. But if the sign of the second term be changed, these values will satisfy it; that is, the roots of $x^2 - \sqrt{(x^2 - 7)} = 5$ are 4 and $\frac{8}{3}$. See also Horner on Surd Equations, *Phil. Mag.*, 1836, vol. viii. p. 43. The subject is treated of in works on the theory of equations, vol. i. p. 199.

110. Divide every term by $x^{\frac{1}{q}}$; then $x^{\frac{q-p}{2pq}} - \frac{1}{2}.\frac{a^2-b^2}{a^2+b^2}.$ $x^{\frac{1}{p}-\frac{1}{q}} - \frac{1}{2}.\frac{a^2-b^2}{a^2+b^2}.$ $1 = 0$, or reducing the fractional indices, and transposing, $x^{\frac{q-p}{2pq}} - \frac{1}{2}.\frac{a^2-b^2}{a^2+b^2}.x^{\frac{q-p}{pq}} = \frac{1}{2}.\frac{a^2-l^2}{a^2+b^2}.$ The higher power being in the second term, the equation must be

reduced; multiply then by $2(a^2+b^2)$; this gives $2(a^2+b^2)$. $x^{\frac{q-p}{2pq}} - (a^2-b^2)\cdot x^{\frac{q-p}{pq}} = a^2-b^2$, or $-(a^2-b^2)\cdot x^{\frac{q-p}{pq}} + 2(a^2+b^2)\cdot x^{\frac{q-p}{2pq}} = a^2-b^2$. Divide now by $-(a^2-b^2)$, then $x^{\frac{q-p}{pq}} - \frac{2(a^2+b^2)}{a^2-b^2}\cdot x^{\frac{q-p}{2pq}} = -1$. Hence Rule I., $x^{\frac{q-p}{pq}} - \ldots + \left(\frac{a^2+b^2}{a^2-b^2}\right)^2 = \left(\frac{a^2+b^2}{a^2-b^2}\right)^2 - 1 = \frac{4a^2b^2}{(a^2-b^2)^2}$. Evolve, and transpose the second term of the root; then $x^{\frac{q-p}{2pq}} = \frac{a^2+b^2}{a^2-b^2} \pm \frac{2ab}{a^2-b^2} = \frac{(a\pm b)^2}{a^2-b^2} = \frac{a\pm b}{a\mp b}$; $\therefore$ &c.

XV. SIMULTANEOUS QUADRATIC EQUATIONS, p. 184.

1. From (1), $4y=2x \therefore y=\frac{1}{2}x$. Hence from (2), $x^3-\frac{1}{8}x^3 = 56$, &c. 2. From (2), $y=\frac{2}{5}x$; $\therefore$ from (1), $\frac{2x^2}{5}+\frac{4x^2}{25}=126$, $\therefore$ &c. 3. From (1), $8(x^2+y^2)=17(x^2-y^2)$; or $25y^2=9x^2$ but from (2), $y^2=\frac{45}{x}$, $\therefore \frac{1125}{x}=9x^2$ and $1125=9x^3$. 4. Take the product of extremes and of means in (1) and then divide by $(\sqrt{x}-\sqrt{y})$, $\therefore \sqrt{x}+\sqrt{y}=8$ and $x+2\sqrt{xy}+y=64$; subtract 4 times (2), or $4\sqrt{xy}=60$, then $x-2\sqrt{xy}+y=4$, taking the root, $\sqrt{x}-\sqrt{y}=\pm 2$; but $\sqrt{x}+\sqrt{y}=8$, $\therefore 2\sqrt{x}=10$ or 6; $\therefore$ &c. 5. Divide (1) by (2), *i.e.*, invert (2) and multiply (1) by it; then $\frac{x^3-y^3}{x^3+y^3}=\frac{7}{9}$; $\therefore 9x^3-9y^3=7x^3+7y^3$; $\therefore x^3=8y^3$ and $x=2y$; hence from (1), $4y^2+2y^2+y^2=21y$, &c.

6. To both sides of (1) add 4; take the square root and transpose, then $xy=8$ or -12; and $4xy=32$ or -48...(3). Square the members of (2) and from this square subtract those of (3); then $x^2-2xy+y^2=84$ or 4; $\therefore x-y=\pm 2$; or $\pm 2\sqrt{21}$, but $x+y=6$, $\therefore$ &c. 7. Clear (1) and add 36 to both sides, take the root and transpose 6; then $3 \cdot \frac{x}{y} = \pm 11 - 6$. This gives $3x-5y=0$; and $3x+17y=0$; from (2), $3x$

$-3y=6$, $\therefore$ &c. 8. Eq. (1) gives $y=8-x$; Eq. (2), $xy-y-2x=2$, $\therefore x(8-x)-(8-x)-2x=2$, or $x^2-7x=-10$; hence &c. 9. Eq. (2) gives, by adding 1, $y^2-2y+1=12-4x$; $\therefore y=1\pm\surd(12-4x)$, $\therefore$ from (1), $x+4\pm4\surd(12-4x)=14$; transpose, sq. both sides, then $x^2+44x=92$; $\therefore$ &c. 10. From the sq. of (1) subtract 4 times (2); then $x^2-2xy+y^2=a^2-4b$, $\therefore x-y=\pm\surd(a^2-4b)$; but $x+y=a$, $\therefore$ &c. 11. From the sq. of (1) subtract (2); then $2xy=a^2-c$, $\therefore 4xy=2a^2-2c$; subtract this from the sq. of (1), $\therefore x^2-2xy+y^2=2c-a^2$, with the root of this combine (1), $\therefore$ &c. 12. Multiply Eq. (1) by 2, and (2) by 5, and add the results, then $27x^2-11y^2=9$; $\therefore y=\sqrt{\frac{27x^2-9}{11}}$; substitute in (1), and $6x^2-5x\sqrt{\frac{27x^2-9}{11}}+\frac{54x^2-18}{11}=12$. Again, multiply Eq. (1) by 3, and (2) by 2 and add; then $24x^2-11xy=30$; $\therefore 24x^2-30=11x\times\sqrt{\frac{27x^2-9}{11}}$. Square both sides of this last, transpose and divide by 9, then $31x^4-149x^2=-100$, &c. Otherwise, thus, for x put zy, then $(6z^2-5z+2)y^2=12\ldots(3)$; $(3z^2+2z-3)y^2=-3\ldots(4)$. Equate the expressions for y^2 from (3) and (4), clear of fractions, incorporate and divide by 9; then $6z^2+z=\frac{10}{3}$, whence z, and the others easily. 13. For x put zy, separate y^2 and equate its two values, then $12(z-2)=1\times(z^2+z)$; $\therefore z^2-11z=-24$, &c. 14. Let $zy=x$; then $z^2y^2+zy^2=144$, and $zy^2-y^2=14$; or $(z^2+z)y^2=144$, and $(z-1)\times y^2=14$; $\therefore \frac{z^2+z}{z-1}=\frac{72}{7}$, which gives $7z^2-65z=-72$, $\therefore z=8$, or $\frac{9}{7}$, $\therefore$ &c. Otherwise, thus, from (1) take (2), from the result get an expression for x, substitute this in (2) and square; then $130y^2-y^4=(14+y^2)^2$; or $y^4-51y^2=-98$, &c. 15. Assume $x=z+v$, $y=z-v$, $\therefore x+y=2z=2b$, $\therefore z=b$; $x=b+v$; $y=b-v$, $\therefore$ from (1), $(b+v)^3+(b-v)^3=a(b+v)(b-v)$; developing and transposing, $(a+6b)v^2=b^2(a-2b)$, whence $v=\pm b\sqrt{\frac{a-2b}{a+6b}}$ and $\therefore$ the others have their expressions above in terms of v. 16. Square Eq. (1) and take from the result 4 times the 4th power of (2), $x^8-2x^4y^4+y^8=4225$; $\therefore x^4-y^4=\pm65$; but $x^4+y^4=97$, $\therefore$ &c. 17. Cube (1) and from the result take (2); then $3(x+y)xy=5940$, or $60xy=5940$, $\therefore$ from (1), $xy=99$, and $x(20-x)=99$; *i.e.*, $x^2-20x=-99$.

Otherwise, divide (2) by (1); the quotient, $x^2 - xy + y^2 = 103$, taken from the square of (1) gives $3xy = 297$; or $xy = 99$; this last taken from the quotient above gives $x^2 - 2xy + y^2 = 4$, &c. 18. From the square of (2) subtract (1); then $2xy = 270$ and $\therefore 4xy = 540$; take this latter from the square of (2); then $x^2 - 2xy + y^2 = 36$, &c. 19. From the 4th power of (1) subtract (2), then $4xy(x^2 + 2xy + y^2) - 2x^2y^2 = 1760$, or by (1), $4\,xy \times 49 - 2x^2y^2 = 1760$, $\therefore\ x^2y^2 - 98xy = -880$, $\therefore\ xy = 88$, or 10. Take the value 10, square (1) and subtract $4xy = 40$, then $x^2 - 2xy + y^2 = 9$, $\therefore\ x - y = \pm 3$. Combine this with (1), $\therefore$ &c. The values found by taking $xy = 88$ are imaginary. They are $x = \frac{1}{2}.(7 \pm \sqrt{-303})$ and $y = \frac{1}{2}(7 \mp \sqrt{-303})$. Otherwise put $x = v + z$ and $y = v - z$ $\therefore\ x + y = 2v = 7 = a$ $\therefore\ v = \frac{a}{2}$. Substituting the assumed values of x and y, (2) becomes $2v^4 + 12v^2 \times z^2 + 2z^4 = 641 = b$, $\therefore\ z^4 + 6v^2z^2 + v^4 = \frac{b}{2}$; put $\frac{a}{2}$ for v, $z^4 + \frac{3}{2}a^2z^2 + \frac{a^4}{16} = \frac{b}{2}$; $\therefore\ z^4 + \frac{3}{2}a^2z^2 = \frac{8b - a^4}{16}$, a quadratic which gives $z^2 = -\frac{3}{4}a^2 \pm \sqrt{\frac{a^4 + b}{2}}$; putting for a and b their values, this becomes $z^2 = -\frac{147}{4} \pm \sqrt{1521}$, that is $z^2 = \frac{9}{4}$ or $-\frac{303}{4}$; $\therefore$ &c. Or thus, since $2v = 7$ and $v = 3\frac{1}{2}$, put this value for v in the equation $2v^4 + 12v^2z^2 + 2z^4 = 641$; then $\frac{2401}{8} + 147z^2 + 2z^4 = 641$, $\therefore 16 \times z^4 + 1176z^2 = 2727$, $\therefore\ z^2 = \frac{9}{4}$ and $z = \frac{3}{2}$, &c. 20. To the double of (2) add (1) and also subtract it, and we get $x^2 + 2xy + y^2 = 169$; $x^2 - 2xy + y^2 = 1$; $\therefore\ x + y = \pm 13$, $x - y = \pm 1$, &c. 21. From (2) $x + y = \frac{120}{xy}$, $\therefore\ x^2 + 2xy + y^2 = \frac{14400}{x^2y^2}$...(3). Again divide $x^3 + y^3$ by $x + y$ and 152 by $\frac{120}{xy}$; then $x^2 - xy + y^2 = \frac{19xy}{15}$...(4); and if this be taken from (3) we get $3xy = \frac{14400}{x^2y^2} - \frac{19xy}{15}$; clear, transpose, and extract the 3rd root then $4xy = 60$ and $x = \frac{15}{y}$, hence from (2), $xy(x + y) = 15\left(\frac{15}{y} + y\right)$

$=120$, and $y^2-8y=-15$ &c. 22. From (3) by substitution in (1) and (2) we have $x+\sqrt{xz}+z=21\ldots(4)$; and $x^2+xz+z^2=189\ldots(5)$. Now the quotient of (5) by (4) is $x-\sqrt{xz}+z=9\ldots(6)$. Add (6) to (4) and $x+z=15\ldots(7)$; $\therefore$ from (1), $y=6$, $\therefore y^2=36$; $\therefore$ from (3) $4xz=4\times 36=144$. Also taking this from the square of (7) we have $x^2-2xz+z^2=81$, $\therefore x-z=\pm 9$, but $x+z=15$, $\therefore$ &c.

23. Assume $x=z+v$, $y=z-v$; $\therefore x+y=2z=10$ and $z=5$. Then $(z+v)^5+(z-v)^5=2z^5+20z^3v^2+10zv^4=17050$. Divide by 2, substitute the value of z; then $3125+1250v^2+25v^4=8525$. Hence $v^4+50v^2=216$; $\therefore v^2=-25\pm 29=4$, or -54, $\therefore v=\pm 2$, or $\pm 3\sqrt{-6}$; but $z=5$, $\therefore x=5\pm 2=7$, or 3, and $y=10-x=3$ or 7. The imaginary values are, $x=5\pm 3\sqrt{6}\sqrt{-1}$; $y=5\mp 3\sqrt{6}\sqrt{-1}$. 24. Take in (1) the $4a$th root, and in (2) the $(x+y)$th root of y, and equate the two values of y thus found. Then $x^{\frac{x+y}{4a}}=x^{\frac{a}{x+y}}$, hence these indices are equal; that is $\frac{x+y}{4a}=\frac{a}{x+y}$, this equation gives $x+y=2a\ldots(3)$; substitute this in (1) and $x^{2a}=y^{4a}$; *i.e.*, $x=y^2$; substitute this in (3); then $y^2+y=2a$, $\therefore$ &c. 25. Multiply (2) by 3, and add the product to (1), then $11x^2-3y^2=41$ and $x^2=\frac{1}{11}(41+3y^2)\ldots(3)$; substitute this in (1), then $\frac{82+6y^2}{11}-3y\sqrt{\left(\frac{41+3y^2}{11}\right)}+3y^2=5$; $\therefore 39y^2+27=33y\sqrt{\left(\frac{41+3y^2}{11}\right)}$; squaring this, $1521y^4+2106y^2+729=4059y^2+297y^4$, $\therefore 136y^4-217y^2=-81$, $\therefore y^4-\frac{217}{136}y^2=-\frac{81}{136}$, $\therefore y^4-\frac{217}{136}y^2+\left(\frac{217}{272}\right)^2=\frac{47089}{73984}-\frac{44064}{73984}=\frac{3025}{73984}$ $\therefore y^2-\frac{217}{272}=\pm\frac{55}{272}$ $\therefore y^2=\frac{217\pm 55}{272}=1$, or $\frac{81}{136}$, $\therefore y=\pm 1$, or $\pm\frac{9}{\sqrt{136}}=\pm 1$, or $\pm\frac{9}{68}\sqrt{34}$, $\therefore$ from (3), $x^2=\frac{44}{11}=4$, or $\frac{1}{11}\cdot\frac{5819}{136}$ $\therefore x=\pm 2$, or $\sqrt{\frac{529}{136}}=\pm 2$, or $\frac{23}{\sqrt{136}}=\pm 2$, or $\pm\frac{23}{68}\sqrt{34}$. Otherwise thus, let $x=yz$ according to the form, p. 179, Ex. 1, and we have from (1), $2y^2z^2-3y^2z+3y^2=y^2(2z^2-3z+3)=5\ldots(3)$; and from (2), $3y^2z^2+y^2z-2y^2=y^2(3z^2+z-2)=12\ldots(4)$. Equating the values of

y^2 deduced from these, we have $\frac{5}{2z^2-3z+3}=\frac{12}{3z^2+z-2}$; clearing and transposing, $9z^2-41z=-46$;

$$\text{whence } z=\frac{41\pm\surd(41^2-4.9.46)}{18};$$

or $z=2$, or $\frac{23}{9}$. Substituting in (3) this value of z, we find $y=\pm1$, or $\pm\frac{9}{68}\surd34$ &c.

26. Assume $x=v+z$, $y=v-z$ $\therefore v=\frac{a}{2}$; then, as in Ex. 23, $2v^5+20v^3z^2+10vz^4=b$; divide by $10v$ and arrange by z; then $z^4+2v^2z^2+\frac{v^4}{5}=\frac{b}{10v}$. Transpose $\frac{v^4}{5}$, complete the square (Rule I.), then $z^4+2v^2z^2+v^4=\frac{b}{10v}-\frac{v^4}{5}+v^4=\frac{b+8v^5}{10v}$; hence $z=\pm\surd\left(-v^2\pm\sqrt{\frac{b+8v^5}{10v}}\right)$; or putting $\frac{a}{2}$ for v, $z=\pm\surd\left(-\frac{a^4}{4}\pm\sqrt{\frac{4b+a^5}{20a}}\right)$. Now $x=\frac{a}{2}+z$, $y=a-x$. Otherwise; involve (1) to the fifth power, subtract (2) from the result and divide by $5xy$; then $x^3+2x^2y+2xy^2+y^3=\frac{a^5-b}{5xy}$...(3); subtract this from the cube of (1), then $(x+y)xy=axy=a^3-\frac{a^5-b}{5xy}$; clearing, $5ax^2y^2=5a^3xy+b-a^5$; transpose, divide by $5a$; then $x^2y^2-a^2xy=\frac{b-a^5}{5a}$; complete the square, &c. $xy=\frac{a^2}{2}\pm\sqrt{\frac{a^5+4b}{20a}}=c$, suppose; $\therefore x=\frac{c}{y}$ and (1), $\frac{c}{y}+y=a$; $\therefore y^2-ay=-c$; hence $y=\frac{a}{2}\pm\surd\left(\frac{a^2}{4}-c\right)=$ &c. (see Ans.) Now, $x=a-y$ $\therefore$ &c. (see Ans.) Otherwise, $(x^5+y^5)\div(x+y)=x^4-x^3y+x^2y^2-xy^3+y^4=\frac{b}{a}$, or $x^4+y^4-xy(x^2+y^2)+x^2y^2=\frac{b}{a}$...(3). But from (1) $x^2+y^2=a^2-2xy$ and $(x^2+y^2)^2=x^4+y^4+2x^2y^2=(a^2-2xy)^2=a^4-4a^2xy+4x^2y^2$, $\therefore x^4+y^4=a^4-4a^2xy+2x^2y^2$. Substituting the values of x^4+y^4 and x^2+y^2

in (3) we find $a^4 - 4a^2xy + 2x^2y^2 - xy(a^2 - 2xy) + x^2y^2 = \frac{b}{a}$; hence $5x^2y^2 - 5a^2xy = \frac{b}{a} - a^4$. Put xy, deduced from this (as above), equal to c; so that $xy = a$, but $x + y = a. \therefore (x + y)^2 - 4xy = a^2 - 4c$; *i.e.*, $(x - y)^2 = a^2 - 4c$; $\therefore x - y = \pm\sqrt{(a^2 - 4c)}$. This combined with $x + y = a$ gives the values required. 27. Involve (1), subtract as before, and divide by xy, then $4x^2 + 6xy + 4y^2 = a^4 - b$. Take this from four times the square of (1), then $2xy = 4a^2 - \frac{a^4 - b}{xy}$; or $2x^2y^2 - 4a^2xy = b - a^4$. Put the value found from this, $xy = c \therefore x = \frac{c}{y}$ and $\frac{c}{y} + y = a, \therefore y^2 - ay = c$, whence y and $\therefore x$, or $a - y$. Otherwise, assume $x = v + z$, $y = v - z$, involve, add, and divide by 2, then transpose v^4, and $z^4 + 6v^2z^2 = \frac{b}{2} - v^4$.

Hence $z = \pm\sqrt{-3v^2 \pm \sqrt{\left(\frac{b^2}{2} + 8v^4\right)}}$; but $v = \frac{a}{2}$, therefore $z = \pm\sqrt{\left(-\frac{3a^2}{4} \pm \sqrt{\frac{b + a^4}{2}}\right)}$. Now, $x = \frac{a}{2} + z$ and $y = \frac{a}{2} - z$; hence the other values. 28. To (1) add the double of (2) and also subtract it; then $x^2 + 2xy + y^2 = 121$; and $x^2 - 2xy + y^2 = 9$, hence $x + y$ and $x - y$. 29. Divide (2) by (1) and take the quotient from the square of (1), then $3xy = 51$, $\therefore xy = 17$ and $4xy = 68$. Take this from the square of (1), hence $x - y = \pm 16$ &c. 30. Let $x = v + z$, $y = v - z$; by substituting these (1) becomes $10z^4 + 20z^3v^2 + 2z^5 = 3093$; but from (2) $z = \frac{3}{2}$; clearing and developing $240v^4 + 1080v^2 + 243 = 49488$; $\therefore 16v^4 + 72v^2 = 3283$; $\therefore$ &c. 31. Here $(\sqrt{x} - \sqrt{y})^2 - (\sqrt{x} - \sqrt{y}) = 0$; complete the square by adding $\frac{1}{4}$ and take the root; then $\sqrt{x} - \sqrt{y} = 1$ or 0; but $\sqrt{x} + \sqrt{y} = 5$ $\therefore$ &c. 32. Assume $x = z + v$, $y = z - v$ $\therefore$ from (1) $z = 2$; also $x^2 + y^2 = (2 + v)^2 + (2 - v)^2 = 8 + 2v^2$ and $x^3 + y^3 = 16 + 12v^2$. Hence $(8 + 2v^2)(16 + 12v^2) = 280$; $\therefore (4 + v^2)(4 + 3v^2) = 35$; that is, $3v^4 + 16v^2 + 16 = 35$, or $3v^4 + 16v^2 = 19$, and $9v^4 + 48v^2 = 57$. Hence $9v^4 + 48v^2 + 64 = 121$. Hence $v = \pm 1$, or $\pm\sqrt{-\frac{19}{3}}$.

Hence easily $x = 3$ or 1, or $3 \pm \sqrt{\left(-\frac{19}{3}\right)}$ and $y = 1$ or 3,

or $2 \mp \sqrt{\left(-\frac{19}{3}\right)}$. 33. Multiply (1) by $x-y$ then $x^2-y^2-\sqrt{(x^2-y^2)}=6$. This quadratic gives $\sqrt{(x^2-y^2)}=3$ or -2, $\therefore x^2-y^2=9$ or 4; but $x^2+y^2=41$; hence x and y easily. 34. Multiply both terms of the first fraction in (1) by $\sqrt{(y^2+1)}-1$, and both terms of the second fraction by $\sqrt{(x+9)}-3$ then $\frac{y}{\sqrt{(y^2+1)}-1}=\frac{x}{\sqrt{x}\sqrt{(x+9)}-3}$. Multiply this by the terms of (1), then y and x disappear as factors of the radical, and we get $\frac{\sqrt{(y^2+1)}+1}{\sqrt{(y^2+1)}-1}=\frac{\sqrt{(x+9)}+3}{\sqrt{(x+9)}-3}$. Employing the artifice of Exs. 3, 11, pp. 139, 141, we obtain $\sqrt{(y^2+1)}=\frac{1}{3}\cdot\sqrt{(x+9)}$, and $\therefore y^2+1=\frac{1}{9}(x+9)$; $\therefore 9y^2=x$. In (2) put this value of x, then $9y^2(y^2+2y+1)=36y^3+64$; transpose $36y^3$, and $9y^2(y^2-2y+1)=64$, which readily gives $y^2-y=\pm\frac{8}{3}$. 35. Let $y=v+z$, $x=v-z$, then (1) becomes $v^2-2vz+9z^2=17$...(3), while (2) becomes $4vz=16$...(4); take (4) from (3) and extract the square root of the answer; this gives $v-3z=\pm 1$...(5). Adding twice (4) to (3), and evolving, $v+3z=\pm 7$...(6). From (5) and (6) $v=\pm 4$, or ± 3, $z=\pm 1$, or $\pm\frac{4}{3}$, hence x and y easily.

XVI. THEORY OF QUADRATIC EQUATIONS, p. 191.

1. Clear, transpose and the quadratic is $p^2q^2-2mpq=n^2-m^2$, which is to be solved with respect to pq. 2. Arrange by the powers of x; $x^2+(a+b-9)x=(a-5)(4-b)$; complete the square and develop; then $x^2+(a+b-9)x+\left(\frac{a+b-9}{2}\right)^2=4a-ab+5b-20+\frac{1}{4}(a^2+2ab+b^2-18a-18b+81)=\frac{1}{4}(a^2-2ab+b^2-2a+2b+1)=\frac{1}{4}(b-a+1)^2$, $\therefore x+\frac{a+b-9}{2}=\pm\frac{b-a+1}{2}$. 3. The expression reduced gives the quadratic $a^2+43a=594$. 4. Here $x^2-5x=6000$. 5. Here $(x-10)(x+9)=0$. 8. Incorporate, and divide by c and we have $x^2-2ax+a^2-b^2$, &c. . . . 11. Here $x^2-18x=-77$, and

two numbers are required whose sum shall be $+18$ and product $+77$; form No. 2, p. 189. 12. Here $x^2 - \frac{21}{110}x = -\frac{1}{110}$; and the numbers must be such that the product with sign changed is $-\frac{1}{110}$, and sum with sign changed is $-\frac{21}{110}$; form No. 2, p. 189. 13. See form No. 3, p. 189. 14. See form No. 4, p. 189. 15. Here $\frac{1}{4}x^2 - 2x = -10$, and $b^2 < 4ac$ $\therefore$ the roots are imaginary. Again, $3 = a$, $8 = b$, $-6 = c$ and $b^2 < 4ac$, &c.; the roots will be found imaginary by solving the equations. 16. Here $b^2 = 961$, $4ac = 120$, $b^2 - 4ac = 841 = 29^2$, and the roots are real.

XVII. PROBLEMS PRODUCING EQUATIONS, p. 198.

1. Let x be the number, then $\frac{x}{2} + \frac{x}{3} = 20$. 2. Let x be the number, then $4x - \frac{x}{2} = 14$. 3. Here $2x + b = a$. 4. Here $2x - a = b$. See Exs. p. 49. 5. Here $\frac{x}{a} + \frac{x}{b} = c$; $\therefore$ $x = \frac{abc}{a+b}$; so that particular solutions will be had by dividing the continual product of the three numbers by the sum of the two first. Thus if the 8th and 9th parts of a number make 68, the number is found by the expression $\frac{8 \cdot 9 \cdot 68}{8+9}$, or $\frac{4896}{17}$, *i.e.*, 288; and this answer is easily verified. 6. Let x be the less then $x + 9$ is the greater; $\therefore$ $4x - (2x + 18) = 14$. If x be the greater then $x - 9$ is the less; $\therefore$ $4(x - 9) - 2x = 14$. 7. Here $11x$ and $17x$ are in the ratio of 11 to 17; $\therefore$ $11x + 17x = 56$. Or, the parts being x and $56 - x$, we may say, as $x : 56 - x :: 11 : 17$, $\therefore$ $17x = 11(56 - x)$. 8. Here $x + (x + 14) = 150$. 9. Plainly $\frac{3}{5}(3x - 4) = 12$. 10. Let $x =$ the share of the third person since the others are expressed in terms of it; then $x + (x + 9) + (x + 9 + 15) = 300$. The general statement of such questions is in the form $3x + 2a + b = c$, where $x = \frac{1}{3} \cdot (c - 2a - b)$, and the other shares are $\frac{1}{3}(c + a - b)$ and $\frac{1}{3}(c + a + 2b)$. (1.) Here the others are expressed in terms of the quantity

of water, so that we put this $=x$; then $x+15=$ wine and $x+15+25=$ spirits. Hence $x+(x+15)+(x+40)=3x+55=100$; $\therefore$ $x=15=$ water, wine $=30$, spirits $=55$. (2.) Here A's part $=x$; $\therefore x+(x+15)+(x+39)=159$, and the parts are 35, 50, and 74. 11. Here $100-\frac{1}{2}x=2x-100$; the general statement is $a-mx=nx-b$, whence $x=\dfrac{a+b}{m+n}$. Thus, find x such that 220 exceeds $11x$ as much as $17x$ exceeds 102; then $x=\dfrac{220+102}{11+17}=11\frac{1}{2}$. Find x such that $12x$ shall exceed 42 as much as 48 exceeds $6x$; then $x=\dfrac{48+42}{6+12}=5$ &c. 12. Here x being the length of the pole, $\dfrac{x}{3}+\dfrac{x}{4}+10=x$. 13. Similarly $\dfrac{x}{10}+\dfrac{x}{20}+\dfrac{x}{30}+\dfrac{x}{40}+\dfrac{x}{50}+\dfrac{x}{60}+302=x$. 14. Here let body $=x$; $\therefore$ head $=9+\frac{1}{2}x$; $\therefore$ by question body $=9+(9+\frac{1}{2}x)=x$, $\therefore$ $x=18+\frac{1}{2}x$; $\therefore$ body $=36$ lbs.; $\therefore$ head $=27$; tail $=9$. 15. See Ex. 3, p. 194; $36x+30x=396$. 16. The space now to be divided is $396-36=360$. So that $36x+30x=360$. Or $36(x+1)+30x=396$. The time is $5\frac{5}{11}$ hours and distance $196\frac{4}{11}+36=232\frac{4}{11}$ miles. 17. If base $=x$, then $x+(x-11)+(x-16)=75$. 18. Here $x+mx=a$. 19. Here $x+mx+nx=a$; whence $x=\dfrac{a}{1+m+n}$. In the numerical Ex. $x=\dfrac{45}{1+3+5}=5$; the other numbers 15 and 25. 20. Here $x+(x+42)=812$. In generalizing this we have for sum a, and for difference b; and the answers are $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$, see note to p. 25 and Exs. on p. 49. Such questions as the following are at once solved by this expression. An excursion train contained 564 persons, and there were 32 more men than women; the numbers are $\frac{1}{2}(564+32)$ and $\frac{1}{2}(564-32)$ or 298 and 266. A cask contains 84 gallons of rum and sherry mixed, and there are 72 gallons more of rum than of sherry; the numbers are 78 rum, and 6 sherry. 21. The ages at first are x and $3x$; after 15 years $x+15$ and $3x+15$; hence $3x+15=2(x+15)$. 22. Let $x=$ the gals. from 3rd pipe per minute; then $x+10$ and $x-5$ are those from the others, and $20\{x+(x+10)+(x-5)\}=820$.

23. Here $19x$ and $15\,(x+9)$ are the prices; $\therefore$ $19x-1=$

15. $(x+9)$; that is, 1/ off the brandy gives the price of the rum; *i.e.* $34 \times 19 - 1 = 43 \times 15$. 24. Suppose the man alone to drink it in x days, in one day he drinks $\frac{1}{x}$, but the woman drinks $\frac{1}{30}$th, $\therefore 12\left(\frac{1}{x}+\frac{1}{30}\right)=1$. 25. According as x is taken for the *less* or the *greater*, we shall have $(34-x)-18:18-x::2:3$; or $x-18:x-16=2:3$. 26. If he works x days, he is idle $30-x$; for the former he receives $2\,x$ shillings, for the latter he forfeits $(30-x)$ sixpences; $\therefore 2x-\frac{1}{2}(30-x)=35$/. For a general statement we have $px-q(n-x)=a$; hence $x=\frac{nq+a}{p+q}$; and $n-x=\frac{np-a}{p+q}$, which will give particular solutions. 27. A can do $\frac{1}{a}$th in 1 day, and $\frac{x}{a}$ths of the work in x days; so of B; $\therefore \frac{x}{a}+\frac{x}{b}=1$, 1 being the whole work. Similarly $\frac{x}{a}+\frac{x}{b}+\frac{x}{c}=1$, &c. Substitute in the general expressions for the particular solutions (see Ans. p. 344). 28. Let x be the number of bees; $\therefore x-\left(\sqrt{\frac{x}{2}}+\frac{8}{9}x\right)=2$. This gives $x-\frac{8}{9}x-\sqrt{\frac{x}{2}}=2$, or $\frac{x}{9}-\sqrt{\frac{x}{2}}=2$. Multiply by $\frac{9}{2}$; then $\frac{x}{2}-\frac{9}{2}\sqrt{\frac{x}{2}}=9$; a quadratic which gives the answer readily. 29. See Ex. 6, p. 195. Here x being the time past 1, we have $12:1::1+x:x$; $\therefore 11x=1^h=60^m$, $\therefore x=5\frac{5}{11}{}^m$ past 1. The minute hand moves over 12 of the hour spaces, while the hour hand moves over one of them; being together at 12, when the minute hand returns there the hour hand will be at 1; and the next concourse will be at $1^h\,5\frac{5}{11}{}^m$. Now, in any time the hour hand passes over $\frac{1}{12}$th the space passed over by the minute hand, and x being the number of minutes past *one* at which the hands are together $\frac{1}{12}\cdot x=\frac{x}{12}$ will be the number of minute divisions that the hour hand has gone forward; so that we have $5+\frac{x}{12}=x$, or $60+x=12x$, $\therefore 11x=60$ and $x=5\frac{5}{11}{}^m$. For the suc-

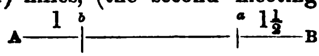
cessive hours 2, 3, 4, &c., the statements will be $12:1::2+x:x$; $12:1::3+x:x$; $12:1::4+x:x$; &c.; the common interval is $1^h\,5\frac{5}{11}{}^m$. 30. Let the capital of the first be x; then $2x-100$, $2(2x-100)-100$ or $4x-300$, and $2(4x-300)-100$, or $8x-700$ are the amounts of the capital at the ends of the first, second, and third years respectively. To find x, put $8x-700=2x$; and $8x-700=\frac{1}{2}x$. 31. Here $50-x=2\times(35-x)$; $\therefore\ x=20$; $\therefore\ 1830-x=1830-20=1810$. 32. The ages are x and $3x$, and $x-7$ and $3x-7$, $\therefore\ 3x-7=5\times(x-7)$. For a general statement, $mx-a=n(x-a)$, $\therefore\ x=\frac{a(1-n)}{m-n}$. 33. Let $2x$ and $3x$ be the miles per hour against and with the stream; the difference x is twice the velocity of the stream. To find the times, divide the spaces by the velocities, and we have $\frac{20}{2x}$ and $\frac{20}{3x}$, $\therefore\ \frac{20}{2x}+\frac{20}{3x}=10$; hence $x=\frac{5}{3}$ and velocity $=\frac{5}{6}$. 34. Let $x=$ the copper, $\therefore\ 100-x=$ the tin; $\therefore\ \frac{21}{4}\cdot x+\frac{17}{4}(100-x)=505$. 35. Let $x=$ value of each in sixpences, then $x-5$ was the value after filing, hence $16(x-5)=8$ gns. 36. Let x and y be the shares; then, first, $x+100=y-100$; second, $y+100=2(x-100)$. 37. Let $x=$ price of port, $y=$ price of sherry; then $20x+30y=120$; and $30x+25y=140$. 38. Let $\frac{x}{y}=$ the fraction; then $\frac{x+4}{y}=\frac{1}{2}$; $\frac{x}{y+7}=\frac{1}{5}$. In general $\frac{x+a}{y}=b$, $\frac{x}{y+c}=d$; $\therefore\ y=\frac{a+dc}{b-d}$, $x=\frac{(a+bc)d}{b-d}$; whence other solutions. 39. Let $x=$ number of guineas, $y=$ number of moidores; then $21x$ and $27y$ are the shillings paid. Hence $x+y=100$ and $21x+27y=£120=2400/.$

40. Let x and y be the time in days in which each could drink the whole, $\therefore\ \frac{1}{x}+\frac{1}{y}=$ consumption of 1 day by both, and $\frac{6}{x}+\frac{6}{y}$ that of 6 days; but $\frac{30}{y}$ is the woman's consumption for 30 days; calling, then, the whole quantity 1,

we have $\frac{15}{x}+\frac{15}{y}=1$; and $\frac{6}{x}+\frac{6}{y}+\frac{30}{y}=1$. 41. Let x and y be the quantities of rye and wheat, then $9x$ and $12y$ are the values in fourpences; also 28 bush. $\times 7=196$, the value of the barley in fourpences, and $3/4 \times 100=10 \times 100=1000$ fourpences, is the value of the mixture; hence the equations are $x+y+28=100$; $196+9x+12y=1000$. 42. Let $x=$ the No. of turkeys, $y=$ No. of geese; then we have $3x$ turkey feathers and y goose feathers; $\therefore\ 3x-2y=15\ldots(1)$. On arriving $(y+10)$ was the No. of geese and $(x-15)$ of turkeys, $\therefore\ y+10:x-15::7:3$, or $3y+30=7x-105\ldots(2)$. Combine this equation with (1). 43. The digits being x and y, the number is $10x+y$; and, calling the quotients q and q', we have $\frac{10x+y}{4}=q+\frac{3}{4}$; $\frac{10x+y}{9}=q'+\frac{8}{9}$. Now by the data $q'=x$, the digit on the left, and $\frac{q}{17}=y$, the digit on the right. Hence the equations are $\frac{10x+y}{4}=17y+\frac{3}{4}$; $\frac{10x+y}{9}=x+\frac{8}{9}$. 44. The equations plainly are $\frac{10y+x}{y-x}=21$; $\frac{10y+x}{y+x}+17=10x+y$. 45. Here the first loss $=\frac{x}{2}$; remainder, $\frac{x}{2}$; he next has $\frac{x}{2}+6$; loss, $\frac{x}{6}+2$; remainder, $\left(\frac{x}{2}+6\right)-\left(\frac{x}{6}+2\right)=\frac{x}{3}+4$. Again he has $\left(\frac{x}{3}+4\right)+12=\frac{x}{3}+16$, $\frac{1}{4}$ of this is $\frac{x}{12}+4$, which taken from $\frac{x}{3}+16$ leaves $\frac{x}{4}+12$; $\therefore\ \frac{x}{4}+12=2$ gns. $=42/$. 46. Plainly here $x(x+16)=960$.

47. Let $x=$ the second number, $y=$ difference of second and least, then the numbers are $x-y$, x, $x+y+5$; $\therefore\ 3x+5=44$, and $x=13$, $\therefore\ (13-y).13.(18+y)=1950$; $(13-y)(18+y)=150$ and $y^2+5y=84$. 48. See Ex. 9, p. 196; $100:x::x:24-x$; $\therefore\ \frac{x^2}{100}=24-x$; $x^2+100x=2400$. 49. Here $x^2+(x+12)^2=2120$. 50. Here $\frac{72}{x}$ is the price of each, the number being x; and $\frac{72}{x+6}=\frac{72}{x}-1$, $\therefore\ x^2+6x=432$. 51. Let $x=$

hypotenuse; then (Euc. I. 47), $x^2=(x-3)^2+(x-6)^2$, or $x^2-18x=-45$. 52. Let $6x$ and $5x$ be the sides, then area $=30x^2$; also area planted $=\frac{1}{6}\cdot 30x^2$; $\therefore\ \frac{1}{6}\cdot 30x^2+625=30x^2$, or $25x^2=625$. 53. Let $x=$ A's rate, $\therefore\ x-1=$ B's; also $\frac{90}{x}=\frac{space}{veloc.}=$ A's time and $\frac{90}{x-1}=$ B's; $\therefore\ \frac{90}{x}+1=\frac{90}{x-1}$. 54. Here $x^2+(x+4)^2=1066$. 55. Let $x=$ No. at first; $\therefore\ x-2=$ No. at last; also £8, 15s. 0d. $=175/$; the shares are $\frac{175}{x}$ and $\frac{175}{x-2}$, and the latter being the greater, we must subtract 10 to make them equal. Hence $\frac{175}{x-2}-10=\frac{175}{x}$. 56. Here $xy=300\ldots(1)$; and $(y+10)(x-8)=300\ldots(2)$; from (1) and (2), $10x-8y=80$, $\therefore\ 10xy-8y^2-80y=0$; but $10xy=3000$ from (1), $\therefore\ 8y^2+80y=3000$. 57. Here, x being the greater, $\frac{2}{3}y+\frac{1}{4}x=7$ and $y-\frac{x}{3}=2$. 58. Let $x=AB$, A′, B′, the persons; then while A′ walks to a, $(x-1\frac{1}{2})$ miles, B′ walks $1\frac{1}{2}$ miles, and on the return while A′ walks $(2x-1)$ miles, (the second meeting being at b,) B′ walks $(x+1)$



miles, and the rates being uniform we have $x-1\frac{1}{2} : 2x-1 :: 1\frac{1}{2} : x+1$, $\therefore\ x=3\frac{1}{2}$. In 1 hr. A′ walks $(1\frac{1}{2}+x-1)=4$, miles, and B′ walks $(x-1\frac{1}{2})+1=3$, miles. 59. Here $x+y=6\cdot\frac{x}{y}$ and $y+x=\frac{3}{2}\cdot\frac{y}{x}$. 60. If $x=$ perp. and $y=$ base, hyp. $=\sqrt{(x^2+y^2)}$, (Euc. I. 47); but area $=\frac{1}{2}\cdot xy$, $\therefore\ \frac{1}{2}xy=\sqrt{(x^2+y^2)}+x+y\ldots(1)$; also, by the data, $x^2+y^2=(x+y)^2-\frac{1}{4}xy^2$. Hence $\frac{1}{4}xy^2=2xy$; and $y=8$. Put this value in (1), then $4x=\sqrt{(x^2+64)}+x+8$. 61. Let $x=$ miles per hr., then $\frac{105}{x}=$ No. of hrs. he travelled, and $\frac{105}{x-2}=$ No. of hrs. if he had travelled slower; $\therefore\ \frac{105}{x}+6=\frac{105}{x-2}$. See Prob. 55. 62. Let x and $x+5$ express the numbers, then smaller flock is worth $xx=x^2$, shillings, and the larger, $(x+5)(x+5)$ shillings; $\therefore\ x^2+x^2+10x+25=$ £65, 13s., 0d. $=1313$ shillings. 63. The stocks are here

x and $416-x$; the gains, $228-x$ and $252-(416-x)=x-164$; but the shares being proportional to the product of the stock and its time in trade, $9x:6(416-x)::228-x:x-164$, and this gives the quadratic $x^2+796x=189696$. 64. Let $x=$ the rate of both since they are always equidistant. The geese travel $1\frac{1}{2}$ miles per hour, and they travel over the five miles interval (50th to 45th mile) in $5\div 1\frac{1}{2}(=\frac{space}{vel.}=t)=\frac{10}{3}$, hours; hence in that time A travels $\frac{10x}{3}$ miles; hence $\frac{10x}{3}-5=$ A's distance from B...(1). Again, since A travels $2x$ miles in 2 hrs. and B $\frac{2x}{3}$ in $40^m=\frac{2}{3}$ hrs. and since they are on opposite sides of the respective milestones we have, $50-2x$ and $31+\frac{2x}{3}$ for the distance from London at which A and B met the waggon, the difference $\frac{8x}{3}-19$ is the distance passed over by the waggon between the meetings; now the rate of the waggon is $2\frac{1}{4}$ miles per hour and time $=\frac{space}{vel.}$. Hence $\left(\frac{8x}{3}-19\right)\div\frac{9}{4}$, or $\frac{4}{9}\left(\frac{8x}{3}-19\right)$, hours is the time elapsed between their meeting the waggon. During this time A has advanced $\frac{4}{9}\left(\frac{8x}{3}-19\right)x$ miles, and the distance between A and B now is $\frac{8x}{3}-19+\frac{4x}{9}\left(\frac{8x}{3}-19\right)$ miles...(2). Equating (1) and (2), since A and B are always equidistant, and making slight reductions we have, $\frac{4x}{9}\left(\frac{8x}{3}-19\right)=\frac{2x}{3}+14$; which finally reduced, is $16x^2-123x=189$. This quadratic gives $x-\frac{123}{32}=\pm\frac{165}{32}$, $\therefore x=9$; hence by (1), $\frac{10x}{3}-5=25$, B's distance from London when A arrived. 65. On the first morning the watch is 11^s behind; two days after, at 9 o'clock, or 45 hours after, it was 2^s behind; hence it gains on the clock 9^s in 45 hours, or $\frac{1}{5}$ of a second per hour. Let now x be the *absolute* gaining rate of the watch per hour, then since $\frac{0{\cdot}1^s}{24}$ is the gaining rate of the clock, $x-\frac{0{\cdot}1^s}{24}$ is the gain

per hour of the watch upon the clock, $\therefore x - \frac{0 \cdot 1^s}{24} = \frac{1}{5}$, $\therefore x = \frac{49}{240}$; and the gaining rate of the watch in 24 hours is 4·9 seconds. This answer is only approximate.

66. Denoting the whole work by 1, and the times by x, y, z, we have $\frac{1}{a} =$ A's work + B's...(1); $\frac{1}{b} =$ A's + C's...(2); $\frac{1}{c} =$ B's + C's...(3); $\therefore \frac{1}{a} - \frac{1}{b} =$ B's − C's...(4). Adding (3) and (4), $\frac{1}{a} - \frac{1}{b} + \frac{1}{c} = 2$ B's $= 2 \cdot \frac{1}{y}$, $\therefore y = \frac{2abc}{ab + bc - ac}$. To find x combine (1) and (3) and to the result add (2). To find z, take (4) from (3); then $\frac{1}{c} + \frac{1}{b} - \frac{1}{a} = 2 \cdot \frac{1}{z} = \frac{2}{z}$, &c. 67. Let $x =$ No. of companies, then $4x =$ No. of men sent from each, hence $4x \cdot x = 4x^2 =$ whole No. of men. Also $3x =$ additional No. furnished, hence $4x^2 + 3x : 4x^2 :: 17 : 16$; $\therefore 68x^2 = 64x^2 + 48x$, &c. 68. Let x and y be the distances travelled by A and B before they meet; then, A's rate : B's :: $x : y$...(1). But since vel. $= \frac{space}{time}$ or $v = \frac{s}{t}$, $v' = \frac{s'}{t'}$, then $v : v' :: \frac{s}{t} : \frac{s'}{t'}$, but if the spaces are equal $v : v' :: \frac{1}{t} : \frac{1}{t'}$, that is $v : v' :: t' : t$; that is the vel. is as the time inversely. Hence we have, A's rate : B's :: $\frac{y}{16} : \frac{x}{36}$...(2). Compounding proportions (1) and (2) we have, (A's rate)^2 : (B's)^2 :: $\frac{xy}{16} : \frac{xy}{36} :: \frac{1}{16} : \frac{1}{36} :: 36 : 16 :: 9 : 4$; $\therefore$ A's rate : B's :: 3 : 2. Again, let $z =$ A's time in hours, then $z + 20 =$ B's time, $(36 - 16 = 20)$. Now, s being given, t is proportional to $\frac{1}{v}$, *i.e.*, the time is inversely as the velocity; hence $z : z + 20 :: 2 : 3$; $\therefore 3z = 2z + 40$, or $z = 40$, and $z + 20 =$ B's time $= 60$.

69. Let $x =$ length of railway in miles; $y =$ diff. of times of starting in hours; then $\frac{x}{36}, \frac{x}{24}\left(= \frac{s}{v} = t\right)$ are the times in which the trains would perform the journey; hence $\frac{x}{36} + y = \frac{x}{24}$ or $\frac{x}{36} = \frac{x}{24} - y$...(1). Again $\frac{x - 12}{36} =$ time by mail train

before collision, and $\frac{2}{3}\cdot\frac{x}{24}$ = time by luggage train before the accident to it; also (by the question), $(\frac{1}{3}x - 12) \div \frac{24}{2}$ or $\frac{x}{36} - 1$ is the time between the accident and collision. Now, deducting the difference of starting we have $\frac{x-12}{36} = \frac{2}{3}\cdot\frac{x}{24} + \left(\frac{x}{36} - 1\right) - y$; or $\frac{x-12}{36} = \frac{x}{18} - 1 - y$...(2). Subtract (2) from (1), then $\frac{1}{3} = 1 - \frac{x}{72}$ and $x = 48$. From (1), $y = 2 - \frac{4}{3} = 40^m$. For a general solution (1) will become $\frac{x}{m} = \frac{x}{n} - y$...(1′); and $\left(\frac{x}{3} - a\right) \div \frac{n}{2} = \frac{2x - 6a}{3n}$; hence $\frac{x-a}{m} = \frac{2}{3}\cdot\frac{x}{n} + \frac{2x-6a}{3n} - y$..(2′). The difference of (2′) and (1′) is $\frac{a}{m} = \frac{x}{3n} - \frac{2x}{3n} + \frac{2a}{n}$; whence $x = 3\left(2 - \frac{n}{m}\right)a$; and $y = \frac{x}{n} - \frac{x}{m} = \frac{m-n}{mn}\cdot x = 3\frac{m-n}{mn}\left(2 - \frac{n}{m}\right) \times a$.

70. Let x = the number of miles from Edinburgh; then $47\frac{1}{2} - x$ = distance from Glasgow; but $11^h\,40^m - 10^h\,30^m = 70^m$, and $11^h\,20^m - 10^h = 80^m$; then $\frac{47\frac{1}{2}}{70}$ and $\frac{47\frac{1}{2}}{80}$ are the rates from either end, the speed being uniform by the question. Also $x \div \frac{47\frac{1}{2}}{70}\left(=\frac{s}{v} = t\right)$ is the time of the Edinburgh train, and $(47\frac{1}{2} - x) \div \frac{47\frac{1}{2}}{80}$ that of the Glasgow train; now the difference in time of starting being 30^m, we have by the data $x \div \frac{47\frac{1}{2}}{70} + 30 = (47\frac{1}{2} - x) \div \frac{47\frac{1}{2}}{80}$. This gives $\frac{70x}{47\frac{1}{2}} + 30 = 80 - \frac{80x}{47\frac{1}{2}}$; $\therefore$ x = distance of point of meeting from Edinburgh = $15\frac{5}{6}$, and distance from Glasgow, $31\frac{2}{3}$ miles.

XVII. INDETERMINATE COEFFICIENTS, p. 214.

1. Assume $\frac{a}{b + cx} = A + Bx + Cx^2 + Dx^3 + \&c.$, then multi-

plying each side by $b+cx$, and transposing a we have, $0 = (Ab - a) + (Bb + Ac)x + (Cb + Bc)x^2 + (Db + Cc)x^3 + \&c.$, $\therefore$ $Ab - a = 0$, $Bb + Ac = 0$, $Cb + Bc = 0$, $Db + Cc = 0$; these give $A = \frac{a}{b}$, $B = -\frac{Ac}{b} = -\frac{ac}{b^2}$; $C = -\frac{Bc}{b} = \frac{ac^2}{b^3}$, &c., $\therefore \frac{a}{b+cx} = \frac{a}{b} - \frac{ac}{b^2}x + \frac{ac^2}{b^3}x^2 - \frac{ac^3}{b^4}x^3 + \&c.$ 2. Assume $\frac{a+bx}{a'+b'x+c'x^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c.$, then multiplying by the denominator, and arranging the terms

$$a + bx = Aa' + \left.\begin{matrix} Ba' \\ +Ab' \end{matrix}\right| x + \left.\begin{matrix} Ca' \\ +Bb' \\ +Ac' \end{matrix}\right| x^2 + \left.\begin{matrix} Da' \\ +Cb' \\ +Bc' \end{matrix}\right| x^3 + \left.\begin{matrix} Ea' \\ +Db' \\ +Cc' \end{matrix}\right| x^4 + \&c.$$

$\therefore Aa' = a$, $A = \frac{a}{a'}$, $Ba' + Ab' = b$, $B = \frac{b}{a'} - A\frac{b'}{a'} = \frac{b}{a'} - \frac{ab'}{a'^2} = \frac{a'b - a'b'}{a'^2}$, $Ca' + Bb' + Ac' = 0$, $C = -B\frac{b'}{a'} - A\frac{c'}{a'} = \frac{ab'^2 - aa'c' - a'bb'}{a'^3}$, &c. 3. Assuming the same series, multiplying by the denominator, arranging by the powers of x, and transposing all the terms to the same side, we have

$$\left.\begin{matrix} A - 1 + B \\ -2Aa \end{matrix}\right| x + \left.\begin{matrix} C \\ -2Ba \\ +A \end{matrix}\right| x^2 + \left.\begin{matrix} D \\ -2Ca \\ +B \end{matrix}\right| x^3 + \left.\begin{matrix} E \\ -2Da \\ +C \end{matrix}\right| x^4 + \&c. = 0$$

$\therefore A - 1 = 0$, $\therefore A = 1$; $B - 2Aa = 0$, $\therefore B = 2a$; $C - 2Ba + A = 0$, $\therefore C = 4a^2 - 1$; $D - 2Ca + B = 0$, $\therefore D = 8a^3 - 2a$; $E - 2Da + C = 0$, $\therefore E = 2a(8a^3 - 2a) - (4a^2 - 1) = 16a^4 - 8a^2 + 1$, &c. 4. In this case there will be no odd powers in the root since its square, $1 + x^2$, contains none, hence we may assume $\sqrt{(1 + x^2)} = 1 + A_2x^2 + A_4x^4 + A_6x^6 + \&c.$ Then squaring both sides and arranging the terms,

$$1 + x^2 = 1 + 2A_2 x^2 + \left.\begin{matrix} 2A_4 \\ +A_2^2 \end{matrix}\right| x^4 + \left.\begin{matrix} 2A_6 \\ +2A_2A_4 \end{matrix}\right| x^6 + \&c.$$

$\therefore 2A_2 = 1$, $A_2 = \frac{1}{2}$, $2A_4 + A_2^2 = 0$, $\therefore A_4 = -\frac{1}{2}A_2^2 = -\frac{1}{2.4}$; $A_6 = -A_2A_4 = \frac{1}{2.2.4} = \frac{1.3}{2.4.6}$, &c. 5. Assume $\sqrt{(1-x)} = A + Bx + Cx^2 + Dx^3 + \&c.$; then squaring and arranging terms &c. $1 - x = A^2 + 2ABx + (2AC + B^2)x^2 + (2AD + 2BC)x^3 + (2AE + 2BD + C^2)x^4 + \&c.$; $A^2 = 1$, $A = 1$; $2AB = -1$, $\therefore B =$

$-\frac{1}{2}$; $2AC+B^2=0$, $\therefore$ $C=-\frac{1}{4\,.\,2}$; $2AD+2BC=0$, $\therefore$ $D=-\frac{2BC}{2A}=-\frac{BC}{A}=-\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2}=-\frac{1\,.\,3}{2\,.\,4\,.\,6}$, &c.

6. Assume here as before, multiply and arrange; then

$$\begin{array}{rl|l|l|l} a= & A+\ \ B & x+\ C & x^2+\ \ D & x^3+\&c. \\ & +2A & +2B & +2C & \\ & & +\ A & +\ B & \end{array}$$

Then $A=a$; $B=-2A=-2a$; $C=-2B-A=4a-a=3a$; $D=-4a$, &c. The same quotient will be found by actual division, Arts. 81, 82, 83. 7. Assume the same series as before, and multiply by $5+7x$, then

$$\begin{array}{rl} 3+2x= & 5A+5Bx+5Cx^2+5Dx^3+\&c. \\ & \quad\ +7Ax+7Bx^2+7Cx^2+\&c. \end{array}$$

Hence, $5A=3$, and $A=\frac{3}{5}$; $7A+5B=2$; $5B=2-\frac{21}{5}$, $B=\frac{2}{5}-\frac{21}{25}=-\frac{11}{25}$; $7B+5C=0$, $\therefore$ $C=-\frac{7}{5}B=\frac{77}{125}=\frac{7\cdot 11}{5^3}$; $5D=-7C=-\frac{7.7.11}{125}$, $\therefore D=-\frac{7.7.11}{125.5}=-\frac{7^2.11}{5^3.5}$; &c. 8. The series here becomes

$$\begin{array}{rl|l|l|l|l} 1+2x= & A+B & x-A & x^2-B & x^3-C & x^4+\&c. \\ & -A & -B & -C & -D & \\ & & +C & +D & +E & \end{array}$$

Then $A=1$; $B-A=2$, $\therefore B=3$; $C=B+A$, $\therefore C=4$; $D=B+C=7$; $E=C+D=11$, &c. 9. The series here is, taking 1 for A, A_1 for B, A_2 for C &c.

$$\begin{array}{rl|l|l|l|l} 1= & 1+A_1 & x+A_2 & x^2+A_3 & x^3+A_4 & x^4+\ \&c. \\ & +\ 1 & +A_1 & +A_2 & +A_3 & \\ & & +1 & +A_1 & +A_2 & \\ & & & +1 & +A_1 & \end{array}$$

Then $A_1=-1$; $A_2+A_1+1=0$, $\therefore A_2=0$; $A_3+A_2+A_1=-1$, $\therefore A_3=0$; $A_4+A_3+A_2+A_1=0$, $\therefore A_4=1$, $\therefore$ &c. 10. Let the given fraction equal $\frac{A}{x-1}+\frac{B}{x+2}$, then $x+3=A(x+2)+B(x-1)$; whence $A+B=1$, $2A-B=3$; $\therefore$ $A=\frac{4}{3}$, $B=-\frac{1}{3}$; or thus: assume as before, and multiply every term by $x-1$; then $\frac{x+3}{x+2}=A+\frac{B(x-1)}{x+2}$. In this put $x-1=0$, $\therefore$ $x=1$, and $\frac{4}{3}=A$. Next multiply each term by $x+2$, then

$\frac{x+3}{x-1} = \frac{A(x+2)}{x-1} + B$. Put $x+2=0$, $\therefore x = -2$, and $-\frac{1}{3} = B$.

11. Here we shall have $\frac{x+1}{x^2-7x+12} = \frac{A}{x-3} + \frac{B}{x-4} = \frac{A(x-4)+B(x-3)}{x^2-7x+12}$, $x+1 = (A+B)x - (4A+3B)$, $\therefore A+B = 1$, $4A+3B = -1$, $\therefore A = -4$, $B = 5$. 12. Assume here for the three component fractions $\frac{A}{x-2}$, $\frac{B}{x-1}$, $\frac{C}{x+1}$, $\therefore$ as before $x^2 = A(x^2-1) + B(x^2-x-2) + C(x^2-3x+2)$, whence $A+B+C=1$...(1). $-B-3C=0$...(2). $-A-2B+2C=0$...(3). From (2) $B=-3C$, $\therefore$ from (3) $8C=A$, and $C=\frac{1}{8}a$, $\therefore$ from (1) $A-3C+C=1$, or $A-2C=1$, *i.e.*, $A-\frac{1}{4}A=1$, and $A=\frac{4}{3}$; $\therefore C=\frac{1}{6}$, and $B = -3C = -\frac{1}{2}$, &c. 13. Let $\frac{1}{(x+a)(x+b)(x+c)} = \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{x+c}$, then $1 = A(x+b)(x+c) + B(x+a)(x+c) + C(x+a) \times (x+b)$. Now, according to the principles of the method this equation is true for any value of x; hence we may put $x = -a$, then $x = -b$, and $x = -c$, on each of which suppositions two terms will vanish; we shall then have $1 = A(a-b)(a-c)$; $\therefore A = \frac{1}{(a-b)(a-c)}$; for $(a-b)(a-c) = (b-a)(c-a)$. $1 = -B \times (a-b)(b-c)$; $\therefore B = -\frac{1}{(a-b)(b-c)}$; for $-b+c = -(b-c)$. $1 = C(a-c)(b-c)$; $\therefore C = \frac{1}{(a-c)(b-c)}$; for $(-c+a)(-c+b) = (a-c)(b-c)$, $\therefore$ &c. 14. Assume $\frac{1}{x^4-a^4} = \frac{A}{x+a} + \frac{B}{x-a} + \frac{C}{x^2+a^2}$; $\therefore 1 = A(x-a)(x^2+a^2) + B(x+a)(x^2+a^2) + C(x+a) \times (x-a)$...(1). Multiplying and arranging the powers, we find

$$1 = \begin{array}{l|l|l|l} -Aa^3 + Ba^3 - Ca^2 + Aa^2 & x - Aa & x^2 + A & x^3 \\ \qquad\qquad\qquad\quad + Ba^2 & \quad + Ba & \quad + B & \\ & \quad + C & & \end{array}$$

$\therefore A+B=0$, $\therefore A = -B$; $-Aa+Ba+C=0$, $\therefore -Aa-Aa+C=0$, $\therefore C=2Aa$; $-Aa^3+Ba^3-Ca^2=1$; $\therefore -Aa^3-Aa^3-2Aaa^2=1$, $\therefore 1 = -4Aa^3$, $\therefore A = -\frac{1}{4a^3}$; $\therefore B = -A = \frac{1}{4a^3}$; $\therefore C =$

$2Aa = -\frac{1}{2a^2}$. Hence $\frac{1}{x^4 - a^4} = -\frac{1}{4a^3(x-a)} + \frac{1}{4a^3(x+a)} - \frac{1}{2a^2(x^2+a^2)}$. Or thus, let $x = -a$ in eqn. (1) above; then $1 = A(-2a)(2a^2) = -4Aa^3 \therefore A = -\frac{1}{4a^3}$. Let $x = a$, $\therefore 1 = B.2a.2a^2, \therefore B = \frac{1}{4a^3}$. Let $x = 0, \therefore 1 = -Aa^3 + Ba^3 - Ca^2, \therefore 1 = \frac{1}{2} - Ca^2, \therefore -\frac{1}{2a^2} = C$; since $\frac{a^3}{4a^3} + \frac{a^3}{4a^3} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. 15. Assume $\frac{1}{x^3 - x^2 - 2x} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x-2}$, then $1 = A(x^2 - 2x) + B(x^2 - x - 2) + C(x^2 + x)$. Arranging the terms, $1 = (A + B + C)x^2 - (2A + B - C)x - 2B$, $\therefore -2B = 1$ and $B = -\frac{1}{2}$; $\therefore A + C = \frac{1}{2}$, also $-2A + C = -\frac{1}{2}, \therefore 3A = 1$ and $A = \frac{1}{3}, \therefore C = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

16. Since $-25 = 5(-5)$, assume $Ax + 5$ and $Bx - 5$ for the factors, then $ABx^2 + (5B - 5A)x - 25$ and $12x^2 + 5x - 25$ are identical. Hence $AB = 12$, $5B - 5A = 5, \therefore B - A = 1$...(1), and $B^2 - 2AB + A^2 = 1$; but $4AB = 48$, $\therefore$ adding this, $B^2 + 2AB + A^2 = 49$; $\therefore B + A = 7$...(2). Eqns. (1) and (2) give $B = 4$, $A = 3$. Hence the factors are $3x + 5$ and $4x - 5$. The same result may be had by solving the quadratic $12x^2 + 5x - 25$; this gives $x = \frac{5}{4}$ or $-\frac{5}{3}$, $\therefore$ the quadratic in its reduced form is $(x - \frac{5}{4})(x + \frac{5}{3})$; or in the above form $(4x - 5)(3x + 5)$, multiplying the one by 4 and the other by 3. 17. Since the only factors of $2x^2$ are $2x$ and x, assume the factors to be $2x + Ay + B$ and $x + ay + b$. These multiplied together become $2x^2 + (2a + A)xy + Aay^2 + (B + 2b)x + (Ba + Ab)y + Bb$; then equating like factors we have $2a + A = -21$...(1); $Aa = -11$...(2). $B + 2b = -1$...(3). $Bb = -3$...(4). $Ba + Ab = 34$. Multiply (1) by a and in the result put for Aa its value in (2), $\therefore 2a^2 + 21a = 11$. This Eq. gives $a = -11$, and $\therefore A = 1$. In the same way we get from (3) and (4), $b = 1$, and $B = -3$.

18. Assume $y = Ax + Bx^3 + Cx^5 + \&c.$

$$\left.\begin{array}{lll} \text{Then since } y = & Ax + Bx^3 + Cx^5 + \&c. & \\ -ay^3 = & & -aA^3x^3 - 3aA^2Bx^5 - \&c. \\ +by^5 = & & +bA^5x^5 + \&c. \\ \&c. & & \&c. \\ -x = -x & & \end{array}\right\} = 0$$

$\therefore$ equating coeffs. of like powers $A - 1 = 0$, $\therefore A = 1$; $B - aA^3 = 0, \therefore B = a$; $C - 3aA^2B + bA^5 = 0, \therefore C = 3a^2 - b, \therefore$ &c.

19. Assume $y = Ax - Bx^2 + Cx^3 - Dx^4 + \&c.$

$$\left.\begin{array}{rl} \therefore -\frac{1}{2}y^2 = & -\frac{1}{2}A^2x^2 + \frac{1}{2}.2ABx^3 - \frac{1}{2}(B^2 + 2AC)x^4 \\ \frac{1}{3}y^3 = & \qquad\qquad \frac{1}{3}.A^3x^3 \\ \&c. & \qquad\qquad \&c. \\ -x = & -x \end{array}\right\} = 0$$

$\therefore A - 1 = 0 \therefore A = 1$; $-B - \frac{1}{2}A^2 = 0$, or $B = -\frac{1}{2}$; also $C + \frac{1}{2}.2AB + \frac{1}{3}A^3 = 0$, $\therefore C + AB + \frac{1}{3}A^3 = 0$, $\therefore C + (-\frac{1}{2}) + \frac{1}{3} = 0$, $\therefore C = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$; $-D - \frac{1}{2}(B^2 + 2AC) = 0, \therefore -D - (\frac{1}{2}B^2 + AC) = 0$ and $-D - (\frac{1}{8} - \frac{1}{6}) = 0$, and $\therefore D + (\frac{1}{8} - \frac{1}{6}) = 0$ and $D = \frac{1}{24}$, &c.

20. Assume $y = Ax + Bx^3 + Cx^5 + \&c.$

$$\left.\begin{array}{rl} \text{Then} \quad y^3 = & \qquad A^3x^3 + 3A^2Bx^5 + 3A^2Cx^7 + \&c. \\ & \qquad\qquad\qquad\qquad + 3AB^2x^7 + \&c. \\ -3y = & -3Ax - 3Bx^3 - 3Cx^5 - 3Dx^7 - \&c. \\ +x = & x \end{array}\right\} = 0$$

Then as before $-3A + 1 = 0$, or $A = \frac{1}{3}$; $A^3 - 3B = 0$ or $B = \frac{A^3}{3} = \frac{1}{3^4}$; $3A^2B - 3C = 0$, or $C = \frac{1}{3^6}$; $\therefore$ &c.

21. Assume $y = Ax + Bx^3 + Cx^5 + \&c.$

$$\left.\begin{array}{rl} \therefore 2y = & 2Ax + 2Bx^3 + 2Cx^5 + \&c. \\ 3y^3 = & \qquad\quad 3A^3x^3 + 9A^2Bx^5 + 9A^2Cx^7 \\ & \qquad\qquad\qquad\qquad + 9AB^2x^7 \\ 4y^5 = & \qquad\qquad\qquad 4A^5x^5 + \&c. \\ \&c. & \qquad\qquad\qquad \&c. \\ -x = & -x \end{array}\right\} = 0$$

$\therefore 2A - 1 = 0$, or $A = \frac{1}{2}$; $2B + 3A^3 = 0$, or $2B = -\frac{3}{8}$, $\therefore B = -\frac{3}{16}$; $2C + 9A^2B + 4A^5 = 0$, or $2C = -4A^5 - 9A^2B = -\frac{4}{32} - 9.\frac{1}{4} \times (-\frac{3}{16}) = -\frac{4}{32} + \frac{27}{64} = \frac{19}{64}$, $\therefore C = \frac{19}{128}$, &c. $\therefore$ &c. 22. Let here $x - 1 = z$ and assume $y = Az + Bz^2 + Cz^3 + Dz^4 +$ &c.

$$\begin{array}{rl} \text{Then } \frac{1}{2}.y^2 = & \frac{1}{2}A^2z^2 + \frac{1}{2}.2ABz^3 + \frac{1}{2}(B^2 + 2AC)z^4 + \&c. \\ \frac{1}{6}.y^3 = & \qquad\qquad \frac{1}{6}A^3z^3 + \frac{3}{6}A^2Bz^4 \ \&c. \\ \frac{1}{24}.y^4 = & \qquad\qquad\qquad \frac{1}{24}A^4z^4 \ \&c. \\ \&c. & \qquad\qquad\qquad \&c. \end{array}$$

$\therefore A = 1$; $B + \frac{1}{2}A^2 = 0$, or $B = -\frac{1}{2}A^2 = -\frac{1}{2}$; $C + \frac{1}{2}.2AB + \frac{1}{6}A^3 = 0$; or $C = -AB - \frac{1}{6}A^3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$; $D + \frac{1}{2}(B^2 + 2AC) + \frac{1}{2}A^2B + \frac{1}{24}A^4 = 0$; or $D + \frac{1}{2}B^2 + AC + \frac{1}{2}A^2B + \frac{1}{24}A^4 = 0$, or $D + \frac{1}{8} + \frac{1}{3} - \frac{1}{4} + \frac{1}{24} = 0$, or $D = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$, &c. $\therefore$ &c.

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. . . 11. Here (Art. 167) the numerators *adf, bcf, bde* of the three ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are represented by 2340, 2310, 2288, the common denominator being 2145, or comparing 1st and 2nd, 2nd and 3rd, 156, 154; 210, 208, the denominators 143, 195 being suppressed. Also the measures are $1\frac{1}{11}, 1\frac{1}{13}, 1\frac{1}{15}$; the first ratio is the greatest, the third the least. 12. Here (Art. 167) $ad = 2a^2 + 24a + 40$, and $bc = 2a^2 + 24a + 64$, and since $bc > ad$, the second ratio is the greatest.

15. By Art. 171 the ratios $a \pm 2x : a$; $a \pm 3x : a$; $a \pm 4x : a$; $a \pm 5x : a$, express nearly the ratios of the 2nd, 3rd, &c., powers, and the ratios $a \pm \frac{1}{2}x : a$; $a \pm \frac{1}{3}x : a$; $a \pm \frac{1}{4}x : a$ &c., nearly those of the 2nd, 3rd, &c., roots. This ratio is the same as 2185 : 2184. . . . 17. According to the supposition let $\frac{a}{x} = n^2$; $\frac{a+y}{x+y} = n$; then $a = n^2x$, $\therefore \frac{n^2x + y}{x+y} = n$, $\therefore n^2x + y = nx + ny$, and $\therefore x(n^2 - n) = y(n-1)$ or $nx(n-1) = y(n-1)$, $\therefore nx = y$ and $\frac{y}{x} = n$. Now $\frac{a}{x} = n^2$, $\therefore \frac{y^2}{x^2} = \frac{a}{x}$, and $y^2x = ax^2$, $\therefore y^2 = ax$ and $y = \sqrt{ax}$, the mean proportional required. . . . 19. For $12 : 24 :: 16 - 12 : 24 - 16 :: 4 : 8$, and $2 : 3 :: \frac{2}{5} : \frac{3}{5}$; $2\frac{2}{5} : 4 :: \frac{3}{5} : 1 :: 3 : 5$, *i.e.*, $12 : 20 :: 3 : 5$. . . . 21. The ratios are $500 + \frac{1}{2} \times 1 : 500 = 1001 : 1000$, $500 + \frac{1}{3} : 500 = 1501 : 1500$. . . . 23. Let $y = mx$, $\therefore 24 = m \,.\, 4$, and $m = 6$; $\therefore y = 6x$. Again $y = mx$, $\therefore 30 = m \,.\, 6$, $\therefore m = 5$, and $y = 5x$. Also $y = mx$, $\therefore a^2 - b^2 = m(a-b)$, $\therefore m = a + b$; and $y = (a + b)x$ is the equation. Lastly, $y = mx$, $\therefore 15 = m \,.\, 3$, $\therefore m = 5$, and $y = 5x$ is the equation. 24. Taking any constant m let $y^2 = m(a^2 - x^2)$...(1), but by the supposition $y - \frac{b^2}{a}$ and $x^2 = a^2 - b^2$, $\therefore$ put these values in (1), $\frac{b^4}{a^2} = m\{a^2 - (a^2 - x^2)\} = mb^2$, $\therefore n^2 = \frac{b^2}{a^2}$ and $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ is the equation required. 25. Here $y = mx$, $\therefore 6 = m \,.\, 4$, $\therefore m = \frac{3}{2}$, $\therefore y = \frac{3}{2} \,.\, 8 = 12$; or, $4 : 8 :: 6 : 12$. 26. Let m and n be the quantities, then $10 = m + n$,

and $y = m \cdot x + n \cdot \frac{1}{x^2}$, by the question. Also, since $y = 13$, $x = 2$, $13 = 2m + \frac{n}{4}$, but $10 = m + n$. These equations give $n = 4$, $\therefore m = 6$, and $\therefore$ the equation required is $y = 6x + \frac{4}{x^2}$. 27. Here $257\frac{1}{3} = mt^2$; but $t = 4$, $\therefore 257\frac{1}{3} = m \cdot 16$, $\therefore m = 16\frac{1}{12}$, and $\therefore s = 16\frac{1}{12} \cdot t^2$ is the equation. 28. Let $x = p \cdot \frac{1}{y^m}$, $y = q \cdot \frac{1}{z^n}$, p and q being constants, then $y^m = q^m \cdot \frac{1}{z^{mn}}$, $\therefore x = pz^{mn} \cdot \frac{1}{q^m}$; $\therefore a = pc^{mn}\frac{1}{q^m}$, $\therefore \frac{p}{q^m} = \frac{a}{c^{mn}}$, $\therefore x = az^{mn} \cdot \frac{1}{c^{mn}}$; or $az^{mn} = xc^{mn}$. 29. Let $y = a + mx + nx^2 \ldots (1)$, and as $x = 1, 2, 3$, $y = 6, 11, 18$, we have

$$6 = a + m + n \ldots (2), \text{ since for } y = 6,\ x = 1$$
$$11 = a + 2m + 4n \ldots (3), \ \ldots\ldots \ y = 11,\ x = 2$$
$$18 = a + 3m + 9n \ldots (4), \ \ldots\ldots \ y = 18,\ x = 3$$
$$\left.\begin{array}{l}\text{from (3) take (2), } 5 = m + 3n \\ \text{from (4) take (3), } 7 = m + 5n\end{array}\right\} \therefore 2 = 2n,\ \therefore n = 1$$

$\therefore m = 2$ and $a = 3$, $\therefore$ (1) becomes $y = 3 + 2x + x^2$. 30. For r put mx, for s, $n\sqrt{x}$, then $y = mx + n\sqrt{x} \ldots (1)$, in which m and n, as in the last question, are to be determined. Then, on the supposition $x = 4$, $y = 5$, (1) becomes $5 = 4m + 2n \ldots (2)$; on the supposition $x = 9$, $y = 10$, (1) becomes $10 = 9m + 3n \ldots (3)$. The Eqns. (2) and (3) give $m = \frac{5}{6}$, $n = \frac{5}{6}$; $\therefore y = \frac{5}{6}(x + \sqrt{x})$. 31. Let $a + b = m(a - b)$, m being constant, then $m = \frac{a + b}{a - b}$, and $\frac{m^2}{1} = \frac{a^2 + 2ab + b^2}{a^2 - 2ab + b^2}$, $\therefore a^2 + 2ab + b^2 : a^2 - 2ab + b^2 :: m^2 : 1$; hence (Art. 183) $2a^2 + 2b^2 : 4ab :: m^2 + 1 : m^2 - 1$, $\therefore a^2 + b^2 = \frac{2(m^2 + 1)}{m^2 - 1} \cdot ab$, so that $a^2 + b^2$ varies as ab, m being constant. 32. Putting v for the volume, we have here $v = mr^3$, $\therefore v = m \cdot 216$. Now, also, $v_1 = m \cdot 27$, $v_2 = m \cdot 64$, $v_3 = m \cdot 125$, $\therefore v_1 + v_2 + v_3 = m(27 + 64 + 125) = m \cdot 216$, $\therefore v = m \cdot 216$ in both cases.

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. . . 31. Here $r = \frac{3}{4}$. 32. $r = \frac{1}{3}$. 33. $r = \frac{2}{3}$. 34. $r = -\frac{2}{3}$.

35. $r=\frac{3}{3}$. 36. $r=\frac{1}{\sqrt{2}}$, and $r=\frac{2}{3}$; then 1st, s or $\left(a.\frac{r^n-1}{r-1}\right)$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\left(\frac{1}{\sqrt{2}}\right)^n-1}{\sqrt{\frac{1}{2}}-1}=\frac{1}{\sqrt{2}}\cdot\frac{\frac{1}{\sqrt{2^n}}-1}{\frac{1}{\sqrt{2}}-1}=\frac{1}{\sqrt{2}}\cdot\frac{\frac{1-\sqrt{2^n}}{\sqrt{2^n}}}{\frac{1-\sqrt{2}}{\sqrt{2}}}=\frac{1}{\sqrt{2}}\cdot\frac{1-\sqrt{2^n}}{\sqrt{2^n}}$$

$\times\frac{\sqrt{2}}{1-\sqrt{2}}=\frac{1}{\sqrt{2^n}}\cdot\frac{1-\sqrt{2^n}}{1-\sqrt{2}}$, dividing both terms by $\sqrt{2}$. Divide both terms by -1, then $s=\frac{1}{\sqrt{2^n}}\cdot\frac{\sqrt{2^n}-1}{\sqrt{2}-1}$. Again $s=\frac{\sqrt{\frac{3}{2}}}{1-\frac{2}{3}}=3\sqrt{\frac{3}{2}}$. . . . 42. Here $s=\frac{n}{2}\{26-(n-1)\frac{1}{3}\}$, &c.

43. Here $d=-\frac{7}{8}$; then $s=\frac{n}{2}\{1+\frac{7}{8}-\frac{7}{8}n)\}$, &c. 44. Here $d=-\frac{1}{n}$; then $s=\frac{n}{2}\left\{\frac{2(n-1)}{n}+(n-1)\times\left(-\frac{1}{n}\right)\right\}=\frac{n}{2}\left(\frac{2(n-1)}{n}-\frac{n-1}{n}\right)=\frac{n}{2}\cdot\frac{n-1}{n}=\frac{n-1}{2}$. 45. The ratio is here $\frac{1}{a-b}$; then

$$s=\frac{a-ar^n}{1-r}=\frac{(a^2-b^2)-(a^2-b^2)\cdot\left(\frac{1}{a-b}\right)^n}{1-\frac{1}{a-b}}=\frac{(a^2-b^2)\left\{1-\frac{1}{(a-b)^n}\right\}}{1-\frac{1}{a-b}}$$

$$=\frac{(a^2-b^2)\cdot(a-b)^n-(a^2-b^2)}{(a-b)^n}\times\frac{a-b}{a-b-1}=\frac{(a+b)(a-b)^n-(a+b)}{(a-b)^{n-1}}$$

$$\frac{a-b}{a-b-1}=\frac{(a+b)(a-b)^n-(a+b)}{(a-b)^{n-2}}\cdot\frac{1}{a-b-1}=\frac{(a+b)}{(a-b)^{n-2}}\cdot\frac{\{(a-b)^n-1\}}{a-b-1}.$$

46. Here $d=-\frac{2}{n}$; then $s=\frac{n}{2}\left\{\frac{2(n-1)}{n}+(n-1)\times\left(-\frac{2}{n}\right)\right\}$ $=\frac{n}{2}\left\{\frac{2(n-1)}{n}-\frac{+2(n-1)}{n}\right\}=0$.

. . . 61. Applying the formula at the end of Art. 216 we have $a=3$, $l=12$, $n=4$, then $\frac{3.36}{3+24}=\frac{108}{27}=4$; $\frac{108}{6+12}=6$. and the means are 4 and 6. Or thus, by Art. 213, $\frac{1}{3}, \frac{1}{x}, \frac{1}{y}, \frac{1}{12}$, are in arithmetic progression, $\therefore a+3d=\frac{1}{12}$, and $3d=\frac{1}{12}-\frac{1}{3}=-\frac{3}{12}$ $\therefore d=-\frac{1}{12}$; $\therefore \frac{1}{3}-\frac{1}{12}=\frac{3}{12}=\frac{1}{4}$; $\frac{1}{4}-\frac{1}{12}=\frac{1}{6}$; so that

the A. P. is $\frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}$, $\therefore$ the H. P. is 3, 4, 6, 12. Again, converting the terms into an A. P. we have $1 + \frac{1}{x} + \frac{1}{y} + \frac{3}{2}$; $\therefore$ $\frac{3}{2} = a + 3d, \therefore d = \frac{1}{6}$; $\therefore$ the A. P. is $1 + \frac{7}{6} + \frac{4}{3} + \frac{3}{2}$; and H. P. is $1 + \frac{6}{7} + \frac{3}{4} + \frac{2}{3}$. The same will be found by substitution as above. Lastly $\frac{(n-1)al}{a+(n-2)l} = \frac{432}{54} = 8$; $\frac{(n-1)al}{2a+(n-3)l} = 12$. 62. $\frac{1}{2} + \ldots + \ldots \frac{1}{12}$ is the A. P., and $\frac{1}{12} = \frac{1}{2} + 5d$; $\therefore d = -\frac{1}{12}$ and the A. P. is $\frac{5}{12}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, &c., and the H. P. $\frac{12}{5}$, 3, 4, 6, &c. Or $\frac{(n-1)al}{a+(n-2)l} = \frac{5.2.12}{2+4.12} = \frac{120}{50} = \frac{12}{5}$, &c. 64. Here $n = 9, \therefore -\frac{1}{2} = 1 + 8d, \therefore d = -\frac{3}{16}$, &c. 66. Here $72 = \frac{n}{2}(36 - 2n)$, and $n^2 - 18n = -72, \therefore n = 12$, or 6. The first six terms make 72, and if we continue the series to 12 terms, the sum is also 72. See Ex. 3, p. 238. 67. The quadratics here are $n^2 + 6n = 40$; $13n^2 + 47n = 396$, and $n^2 + 6n = 567$. The negative values of n are satisfied by extending the series to *the* left of the first term; thus in the first, $7 + 9 + 11 + 13 = 40$; also $-13 - 11 - 9 - 7 - 5 - 3 - 1 + 1 + 3 + 5 = 40$ &c. 70. See form. (3) end of Art. 215. 71. See Art. 208. 72. See (9) Art. 208. 73. Here by (11) Art. 208, or from the quadratic $3n^2 - 111n = -1026$, *i.e.* $n^2 - 37n = -342$, n has two positive values, see Ex. 3, p. 238. 74. The quadratic here is $n^2 + 11n = 962$. 76. See Ex. 70 above. 79. The quadratic is $n^2 - 20n = -91$. 81. See Ex. 4. p. 240. 82. Here 5th term is $a + 4d = 13$, the 9th $a + 8d = 25$; $\therefore d = 3$ $\therefore a = 1$, and 7th term is 19. 83. Here $d = \cdot 0004$; $2s = 5\cdot 496$; then $2\cdot 748 = \frac{n}{2}(\cdot 068 + \overline{n-1} \times \cdot 0004)$; hence $\cdot 0004n^2 + \cdot 0676n = 5\cdot 496$; divide by $\cdot 0004$; and $n^2 + 169n = 13740$.

84. By hypothesis, and dividing the first by the second term $\frac{2\sqrt{xy}}{x+y} = \frac{b}{a}$ and $\frac{4xy}{(x+y)^2} = \frac{b^2}{a^2}$; hence $1 - \frac{4xy}{(x+y)^2} = 1 - \frac{b^2}{a^2}$ and clearing of fractions, $\frac{x^2 - 2xy + y^2}{x^2 + 2xy + y^2} = \frac{a^2 - b^2}{a^2}$; $\therefore \frac{x-y}{x+y} = \frac{1}{a} \times \sqrt{(a^2 - b^2)}$. Multiplying both terms by $\frac{x}{y} + 1$ and by a we have $a\frac{x}{y} - a = \sqrt{(a^2 - b^2)}\frac{x}{y} + \sqrt{(a^2 - b^2)}$, transposing, $a - \sqrt{(a^2 - b^2)}$

$\frac{x}{y} = a + \sqrt{(a^2 - b^2)}$, hence $\frac{x}{y} = \frac{a + \sqrt{(a^2 - b^2)}}{a - \sqrt{(a^2 - b^2)}}$; $\therefore$ &c. 85. Since $\frac{a}{a} = 1$, and $\frac{a - b}{b - c} = 1$, $\therefore a - b = b - c$; hence Art. 202, a, b, c are in A. P. Again if $\frac{a - b}{b - c} = \frac{a}{b}$, then $ab - b^2 = ab - ac \therefore b^2 = ac$, and a, b, c, are in G. P. Lastly, if $\frac{a - b}{b - c} = \frac{a}{c}$, $ac - bc = ab - ac$ or $b(a + c) = 2ac$ and $b = \frac{2ac}{a + c}$, that is b is the H. M. and a, b, c are in H. P. Otherwise, from the supposition $a : c :: a - b : b - c$; $\therefore$ &c. 86. Since a, b, c are in H. P., $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P.; hence $\frac{a + b + c}{a}, \frac{a + b + c}{b}, \frac{a + b + c}{c}$ are also in A. P. and also $\frac{a + b + c}{a} - 1, \frac{a + b + c}{b} - 1, \frac{a + b + c}{c} - 1$, are in A. P., that is $\frac{b + c}{a}, \frac{a + c}{b}, \frac{a + b}{c}$ are in A. P.; and $\therefore$ $\frac{a}{b + c}, \frac{b}{a + c}, \frac{c}{a + b}$ are in H. P.

87. Multiply the A. M. by the H. M. then $\frac{a + b}{2} \times \frac{2ab}{a + b} = ab = \sqrt{ab} \times \sqrt{ab}$. Hence, Art. 178, $\frac{a + b}{2} : \sqrt{ab} :: \sqrt{ab} : \frac{2ab}{a + b}$ Again, since $(a - b)^2$ is always positive; and $a^2 - 2ab + b^2 > 0$, ad l $4ab$, then $a^2 + 2ab + b^2 > 4ab$, $\therefore a + b > 2\sqrt{ab}$, and $\frac{a + b}{2} > \sqrt{ab}$; that is the A. M. > G. M. Next, since $a + b > 2\sqrt{ab}$, $(a + b)\sqrt{ab} > 2\sqrt{ab} \cdot \sqrt{ab} > 2ab$; divide by $(a + b)$; then $\sqrt{ab} > \frac{2ab}{a + b}$; that is G. M. > H. M., much more then is A. M. > H. M.

88. Here $\frac{a + b}{2} = 3$ and $\frac{2ab}{a + b} = \frac{8}{3}$; hence $a + b = 6$ and $\frac{2ab}{6} = \frac{8}{3}$; and $ab = 8 \therefore b = \frac{8}{a}$; and $\therefore a + \frac{8}{a} = 6$ or $a^2 - 6a = -8 \therefore a = 4$ or 2. 89. By hypothesis $\frac{a + b}{2} = 2\sqrt{ab}$; $\therefore \frac{a + b}{2\sqrt{ab}} = 2$; hence

(Art. 183) and Exs. 3 and 11, pp. 139, 141, $\dfrac{a+b+2\sqrt{ab}}{a+b-2\sqrt{ab}}=\dfrac{2+1}{2-1}=\dfrac{3}{1}$; by evolution, $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}=\dfrac{\sqrt{3}}{1}$, $\therefore\ \dfrac{\sqrt{a}}{\sqrt{b}}=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$; squaring this we get $\dfrac{a}{b}=\dfrac{4+2\sqrt{3}}{4-2\sqrt{3}}=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$; &c. 90. As in last Ex. $\dfrac{a+b}{2}=m\dfrac{2ab}{a+b}$, $\therefore \dfrac{1}{4ab}\cdot(a^2+2ab+b^2)=m$; hence (Art. 183.) $\dfrac{a^2+2ab+b^2}{a^2-2ab+b^2}=\dfrac{m}{m-1}$ and $\dfrac{a+b}{a-b}=\dfrac{\sqrt{m}}{\sqrt{(m-1)}}$; $\therefore\ \dfrac{a}{b}=\dfrac{\sqrt{m}+\sqrt{(m-1)}}{\sqrt{m}-\sqrt{(m-1)}}$; &c. 91. Here $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, are in A. P., $\therefore d=\frac{1}{6}$; hence the terms before $\frac{1}{2}$ are $\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$ and $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$; those after $\frac{1}{6}$ are $0, -\frac{1}{6}$; hence the H. terms are $\frac{6}{5}, \frac{3}{2}, \frac{1}{0}, -6$; also the A. P. is $\frac{17}{5}, \frac{10}{5}, \frac{3}{5}$, $\therefore d=\frac{7}{5}$, and the two terms at the beginning, are $\frac{24}{5}, \frac{31}{5}$, at the end $-\frac{4}{5}, -\frac{11}{5}$. Hence the H. terms are $\frac{5}{31}$ and $\frac{5}{24}$; $-\frac{5}{4}, -\frac{5}{11}$. 92. Here $r=-\dfrac{2}{5}$; and $s=\dfrac{5}{2}\cdot\dfrac{1-\left(-\frac{2}{5}\right)^n}{\frac{7}{5}}=$

$$=\frac{5}{2}\cdot\frac{5}{7}\cdot\left\{1-\frac{-2^n}{5^n}\right\}=\frac{5^2}{14}\cdot\frac{5^n-(-2^n)}{5^n}$$

$$=\frac{1}{14}\cdot 5^2\cdot\frac{5^n-(-2^n)}{5^n}=\frac{1}{14}\cdot\frac{5^n-(-2^n)}{5^{n-2}}$$

Again, $r=-\dfrac{a}{x^{\frac{3}{2}}}$ and

$$s=\frac{x^{\frac{5}{2}}\left\{1-\left(-\frac{a}{x^{\frac{3}{2}}}\right)^n\right\}}{1-\left(-\frac{a}{x^{\frac{3}{2}}}\right)}=\frac{x^{\frac{5}{2}}\left\{\frac{x^{\frac{3n}{2}}\mp a^n}{x^{\frac{3n}{2}}}\right\}}{\frac{x^{\frac{3}{2}}+a}{x^{\frac{3}{2}}}}=\frac{x^{\frac{8}{2}}}{x^{\frac{3n}{2}}}\left\{\frac{x^{\frac{3n}{2}}\mp a^n}{x^{\frac{3}{2}}+a}\right\}=$$

$$\frac{1}{x^{\frac{3n-8}{2}}}\cdot\left\{\frac{x^{\frac{3n}{2}}\mp a^n}{x^{\frac{3}{2}}+a}\right\}.$$

93. There are here $n+2$ terms; then (Art. 208) $31=1+(n+2-1)d=1+(n+1)d$, $\therefore d=\dfrac{30}{n+1}$; hence the 8th term or 7th mean is $1+7\dfrac{30}{n+1}=\dfrac{211+n}{n+1}$; also the $(n-1)$th term is $1+\dfrac{30(n-1)}{n+1}=\dfrac{31n-29}{n+1}$; hence by the question $\dfrac{211+n}{n+1}$

$\frac{31n-29}{n+1}::5:9$, or $211+n:31n-29::5:9$, and $2044=146n$, &c. 94. Here $r=\frac{1}{2}\div\frac{1}{2-\sqrt{2}}=\frac{\sqrt{2}-1}{\sqrt{2}}$; $\therefore s=\frac{\sqrt{2}+1}{\sqrt{2}-1}\div\left(1-\frac{\sqrt{2}-1}{\sqrt{2}}\right)=\frac{\sqrt{2}+1}{\sqrt{2}-1}\div\frac{1}{\sqrt{2}}=\frac{2+\sqrt{2}}{\sqrt{2}-1}=4+3\sqrt{2}$. Again $r=a^q, \therefore s=a^p.\frac{a^{pq}-1}{a^q-1}$. 95. Here (Art. 212) $b=\frac{2ac}{a+c}$; $\therefore a^2+c^2-2b^2=a^2+c^2-2\left(\frac{2ac}{a+c}\right)^2$; but $a^2+c^2=(a-c)^2+2ac$; $\therefore$ $a^2+c^2-2b^2=(a-c)^2+2ac-\frac{2(4a^2c^2)}{(a+c)^2}=(a-c)^2+2ac-\frac{2ac(4ac)}{(a+c)^2}=(a-c)^2+2ac\left(1-\frac{4ac}{(a+c)^2}\right)=(a-c)^2+2ac\times\left(\frac{a^2+2ac+c^2-4ac}{(a+c)^2}\right)=(a-c)^2+2ac\frac{(a-c)^2}{(a+c)^2}$. Now, this is a positive quantity, and $\therefore a^2+c^2-2b^2$ is also positive; so that $a^2+c^2>2b^2$. 96. Let the terms be $\frac{1}{2}$, x, y, then $2, \frac{1}{x}, \frac{1}{y}$, form an A. P.; then $\frac{1}{2}+x+y=1\frac{1}{12}\ldots(1)$; also from the A. P. $2, \frac{1}{x}, \frac{1}{y}$, we have (Art. 203) $\frac{2}{x}=2+\frac{1}{y}$; $\therefore \frac{1}{x}=1+\frac{1}{2y}\ldots(2)$. From (2), $y=\frac{x}{2(1-x)}\ldots(3)$; from (1), $x+\frac{x}{2(1-x)}=\frac{7}{12}\ldots(4)$; (4) cleared gives $12x^2-25x=-7$, whence $x=\frac{1}{3}$ or $\frac{7}{4}$; from (3) $y=\frac{1}{4}$ or $-\frac{7}{6}$; then from the A. P., $2, \frac{1}{x}, \frac{1}{y}$, we have the H. P. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. 97. Let a be the 1st term, and d the difference; then the $(p+q)$th term $=a+(p+q-1)d=m\ldots(1)$, and the $(p-q)$th term $=a+(p-q-1)d=n\ldots(2)$; hence $\frac{1}{2}(m+n)=a+(p-1)d=p$th term. Again from (1) and (2) $m-n=2qd$; $\therefore$ $d=(m-n).\frac{1}{2q}$; $\therefore pd=(m-n)\frac{p}{2q}$, now (1) is the same as $a+pd+(q-1)d$; $\therefore a+(q-1)d=m-pd=m-(m-n)\frac{p}{2q}$; but $a+(q-1)d$ is the qth term; hence $m-(m-n)\frac{p}{2q}=q$th *term; and as above* $\frac{1}{2}(m+n)=p$th term.

98. Here the $(p+q)$th term $= ar^{p+q-1} = m$, (1)
$(p-q)$th term $= ar^{p-q-1} = n$, (2).
Hence $mn = a^2r^{2p-2}$; $\therefore \sqrt{mn} = ar^{p-1} = p$th term. Again, $\frac{m}{n} = r^{2q}$; $\therefore r = \left(\frac{m}{n}\right)^{\frac{1}{2q}} \therefore r^p = \left(\frac{m}{n}\right)^{\frac{p}{2q}} \therefore \frac{1}{r^p} = \left(\frac{n}{m}\right)^{\frac{p}{2q}}$. Hence, from (1) $\frac{m}{r^p} = \frac{ar^{p+q-1}}{r^p} = ar^{q-1}$; but this is the qth term; $\therefore$ qth term $= ar^{q-1} = \frac{m}{r^p} = m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$, and as above the pth term $= ar^{p-1} = \sqrt{mn}$. 99. Let A = 1st term, R = com. ratio; then $a = AR^{p-1}, b = AR^{q-1}, c = AR^{r-1}$; $\frac{a}{b} = R^{p-q}$; $\therefore R = \left(\frac{a}{b}\right)^{\frac{1}{p-q}} \ldots (1)$. Again $\frac{b}{c} = R^{q-r} = \left(\frac{a^{q-r}}{b^{q-r}}\right)^{\frac{1}{p-q}}$; $\therefore \frac{a^{q-r}}{b^{q-r}} = \frac{b^{p-q}}{c^{p-q}}$; then clearing $b^{p-r} = a^{q-r}\,c^{p-q}$; and multiplying both sides by b^{r-p}, $1 = a^{q-r} \times b^{r-p} \times c^{p-q}$. Again, if a, b, c were the pth, qth, and rth terms of an A. P., we should have the following equations to determine the unknowns, which are the first term and com. diff.; $a = A + (p-1)\,D$; $b = A + (q-1)\,D$; $c = A + (r-1)\,D$; hence $a - b = (p-q)\,D$; $b - c = (q-r)\,D$, and $\therefore \frac{a-b}{b-c} = \frac{p-q}{q-r}$; therefore $(q-r)\,a - (q-r)\,b = (p-q)\,b - (p-q)c$; combining the terms containing b, cancelling bq, and transposing, we have $(q-r)\,a + (r-p)\,b + (p-q)\,c = 0$. But if a, b, c be the terms of an H.P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the terms of an A. P.; substituting these in the foregoing we have $(q-r)\frac{1}{a} + (r-p)\frac{1}{b} + (p-q)\frac{1}{c} = 0$; which cleared becomes $(q-r)\,bc + (r-p)\,ac + (p-q)\,ab = 0$; which expresses the relation required. 100. Here, a being in succession 1, 2, 3...p, and d being 1, 3, 5... $2p-1$, Art. 50 and 206 (5), we have $s_1 = \frac{n}{2}\{2 + (n-1)\cdot 1\} \ldots (1)$ $s_2 = \frac{n}{2}\{4 + (n-1)3\} \ldots (2)$, $s_3 = \frac{n}{2}\{6 + (n-1)5\} \ldots (3)$, &c. &c. $s_p = \frac{n}{2}\{2p + (n-1)(2p-1)\} \ldots (4)$, $\therefore s_1 + s_2 + s_3 \ldots + s_p = \frac{n}{2}\{(2 + 4 + 6 + \ldots + 2p) + (n-1)[1 + 3 + 5 + \ldots + (2p-1)]\} =$

$\frac{n}{2}\left\{(2+2p)\frac{p}{2}+(n-1)\left[(1+2p-1)\frac{p}{2}\right]\right\}=(p+1+np-p)\frac{np}{2}=(np+1)\frac{np}{2}$. Or thus; when developed (1) is the same as $\frac{n^2}{2}+\frac{n}{2}$, (2) is the same as $\frac{3n^2}{2}+\frac{n}{2}$, (3) is the same as $\frac{5n^2}{2}+\frac{n}{2}$, (4) is the same as $\frac{(2p-1)n^2}{2}+\frac{n}{2}$. The sum $\therefore$ as above to p terms will be $n^2\left(\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\ldots\frac{2p-1}{2}\right)+p\cdot\frac{n}{2}$ or $\frac{n^2}{2}\{1+3+5+\ldots+(2p-1)\}+p\cdot\frac{n}{2}$ or (Art. 206) (5), $\frac{n^2}{2}\{2+(p-1)2\}\frac{p}{2}+\frac{np}{2}$, or, reduced $\frac{np}{2}(np+1)$

XXII. PROBLEMS ON PROGRESSION AND VARIATION, p. 250.

1. Here $x+xy=28$, $xy^2+xy^3=y^2(x+xy)=252$; $\therefore$ $y^2=9$ and $y=\pm 3$, $\therefore x=7$, &c. 2. Let xy^2, xy, x, be the shares; then $xy^2=x+90\ldots(1)$; and since $xy^2+xy+x=210$, $2x+90+xy=210$, $\therefore 2x+xy=120\ldots(2)$; and from (1), $x=\frac{90}{y^2-1}$, from (2), $x=\frac{120}{y+2}$; $\therefore \frac{4}{y+2}=\frac{3}{y^2-1}$, $\therefore 4y^2-3y=10$; whence $y=2$ and $\therefore$ $x=30$; $\therefore$ &c. Or thus, $xy^2+xy+x=210$; but $xy^2-x=90$; $\therefore (y^2-1)x=90$, and $x=\frac{90}{y^2-1}$, whence substituting, $\frac{90y^2}{y^2-1}+\frac{90y}{y^2-1}+\frac{90}{y^2-1}=210$; and $4y^2-3y=10$. Or thus, let $x=$ share of last, $\therefore x+90=$ share of first, $\therefore$ share of second is a mean proportional between them, or $210-(90+2x)=120-2x$, is a mean, that is $x:120-2x::120-2x:x+90$; $\therefore x^2-190x=-4800$, and $x=30$, or -160; &c. 3. Here $x+xy+xy^2=14$, and $2+2y=y+y^2$, $\therefore y^2-y=2$; $\therefore y=2$ and $\therefore$ $x=2$; &c. 4. $x+xy+xy^2+xy^3=30\ldots(1)$, and $\frac{xy^3}{xy+xy^2}=\frac{4}{3}\ldots(2)$; from (2) $\frac{y^2}{1+y}=\frac{4}{3}$, $\therefore y=2$; $\therefore$ from (1) $15x=30$, and $x=2$, &c. 5. $x+(x+y)+(x+2y)=21\ldots(1)$, and $2x+y:2x+$

$3y :: 3 : 4 \ldots(2)$ &c. From (1), $3x + 3y = 21$ or $x + y = 7$; from (2), $8x + 4y = 6x + 9y$; or $2x = 5y \therefore y = \frac{2}{5}x$; and $x + \frac{2}{5}x = 7, \therefore x = 5$, &c. 6. $x + xy = 9, \therefore x = \frac{9}{1+y} \therefore \frac{9}{1+y} + \frac{9y^2}{1+y} = 15 \therefore 9y^2 - 15y = 6; \therefore y = 2$ and $\therefore x = 3$, &c. 7. In 12^h the strokes $=$ the sum of the series $1 + 2 + 3$ &c. to 12 terms $= 78; \therefore$ in 24^h, $78 \times 2 = 156$; and in 30 days the $156 \times 30 = 4680$ strokes. 8. Here the numbers are x, xy, xy^2, then $x^3y^3 = 64 \ldots(1)$, $x^3 + x^3y^3 + x^3y^6 = 584 \ldots(2)$; from (1), $y^3 = \frac{64}{x^3}, \therefore y^6 = \frac{4096}{x^6}$; substituting this value and (1) in (2), and clearing we find $x^6 - 520 \times x^3 = -4096$. Whence $x = 2; \therefore y = 2$. 9. Let the sides be $6 - x, 6, 6 + x \therefore$ (Euc. I. 47), $36 - 12x + x^2 + 36 = 36 + 12x + x^2$; $\therefore 24x = 36, \therefore x = \frac{3}{2}$. But if we call one side 6 and the others $6 + x$, $6 + 2x$, then $36 + 24x + 4x^2 = 36 + 36 + 12x + x^2$, or $x^2 + 4x = 12 \therefore x = 2$, or -6. The sides $\therefore$ are 6, 8, 10; the negative result being here inadmissible. The ambiguity arises from the data of the problem not being sufficiently limited; the algebraic language is more precise than that of the problem. 10. Let the extremes be x and $y \therefore \sqrt{xy}$ will be the mean, then $x + \sqrt{xy} + y = 21 \ldots(1)$; and $x^2 + xy + y^2 = 189 \ldots(2)$; divide (2) by (1), $x - \sqrt{xy} + y = 9 \ldots(3)$; from (3) and (1), $x + y = 15 \ldots(4)$; subtract, $\sqrt{xy} = 6, \therefore xy = 36 \ldots(5)$. Hence $(x + y)^2 - 4xy = 225 - 144 = 81, \therefore x - y = 9; \therefore$ from (4), $x = 12$; from (5), $y = 3, \therefore \sqrt{xy} = 6$. Otherwise, let $\frac{x}{y}, x$, and xy be the quantities, then $\frac{x}{y} + x + xy = 21 \ldots(1)$; $\frac{x^2}{y^2} + x^2 + x^2y^2 = 189 \ldots(2)$; from (1), $xy + \frac{x}{y} = 21 - x; \therefore \frac{x^2}{y^2} + 2x^2 + x^2y^2 = 441 - 42x + x^2, \therefore \frac{x^2}{y^2} + x^2 + x^2y^2 = 21^2 - 42x; \therefore$ from (2), $189 = 21^2 - 42x, \therefore 9 = 21 - 2x, \therefore 2x = 12$, and $x = 6$; substitute this value in (1), then after reduction the quadratic, $2y^2 = -5y - 2$, will give $y = 2, \therefore$ &c.

11. Let $d = 2y$; then the numbers will be $x + 3y$, $x + y$, $x - y$, $x - 3y$; $\therefore$ their sum $= 4x = 32, \therefore x = 8$; also sum of squares is $4x^2 + 20y^2 = 276$, or $20y^2 = 20, \therefore y = 1$, &c. 12. It is required here to sum the series, $s = \frac{47}{2}(2 + 46)$; this is £56, 8*s*. If the successive prices are 1, 3,

5, &c., then $s = \frac{47}{2}(2 + 92) = £110, \ 9s.$ 13. Here £100 = 24,000$d.$, $\therefore$ Art. 208 (9), $n - 1 = \frac{24000 - 6}{3} = 7998$, $\therefore$ $n = 7999$ = number of acres.

14. We have now to sum the two series, $3 + 5 + 7 +$ &c., and $4 + 6 + 8 +$ &c., to x terms each, and to put the sum of the two equal to 168, since the whole distance is travelled by the two. The sums are $\frac{x}{2}(4 + 2x)$ and $\frac{x}{2}(6 + 2x)$. The sum of these is $x(5 + 2x) = 168$, a quadratic giving $x = 8$. Again, if x be the number of days, then the first goes in all $20x$ miles; and the second, $\frac{x}{2}(1 + x)$ miles, $\therefore \frac{x^2 + x}{2} + 20x = 23661$, a quadratic giving $x = 198$ days. 15. Let the No. of days be $x + 5$, and x; the second travels $12x$ miles; the first travels a distance expressed by the sum of the series, $1 + 2 + 3 +$ &c., to $x + 5$ terms; this is $= \frac{x + 5}{2}(x + 6)$. Then by the question, $12x = \frac{x + 5}{2}(x + 6)$, $\therefore$ $x = 3$ or 10; and the distances are 36 and 120. 16. Let x = No. of hours, then since A had gone 11 miles before B started, A's whole distance is $4x + 11$; also, B's distance is $\frac{x}{2}\left(9 + \overline{x - 1} \cdot \frac{1}{4}\right)$; $\therefore$ $4x + 11 = \frac{x}{2}(8\frac{3}{4} + \frac{1}{4}x)$, and $x = 8$ or -11, &c. 17. Let a, b, c, be the quantities, then $a + b + c = 13 \ldots (1)$, and $ac = 18 \ldots (2)$; and Art. 212 (1), $2ac = (a + c)b$, that is, $36 = (13 - b)b = 13b - b^2$, $\therefore$ $b^2 - 13b = -36$, $\therefore$ $b = 4$ or 9; take 4, then from (1), $a + c = 9$, and, combining this with (2), we get $a = 6$, $c = 3$. 18. Here $\frac{a + b}{2} = \sqrt{ab} + 5 \ldots (1)$, and $\frac{2ab}{a + b} = \sqrt{ab} - 4 \ldots (2)$. Multiply (1) by (2), then $ab = ab + \sqrt{ab} - 20$, $\therefore$ $\sqrt{ab} = 20$, and $ab = 400 \ldots (3)$. From (1) take (2), then $\frac{(a + b)^2}{2(a + b)} - \frac{4ab}{2(a + b)} = 9$, clearing, $(a + b)^2 - 4ab = 18(a + b)$, $\therefore$ $(a = b)^2 - 18 + (a + b) = 4ab = 1600$, by (3), $\therefore$ $(a + b)^2 - 18(a + b) = 1600$; complete the square and evolve, $a + b = 50 \ldots (4)$; $\therefore$ $a^2 + 2ab + b^2 = 2500$; subtract 4 times (3), $4ab = 1600$, $\therefore$ $a - b = 30 \ldots (5)$

Combine (4) and (5), $\therefore$ &c. 19. Let a, $\frac{2ac}{a+c}$, c, be the numbers; then by the question $2a+c=\frac{6ac}{a+c}$...(1), and $a^2+c^2=180$...(2); clearing (1), $2a^2+c^2-3ac=0$...(3). Assume $c=ba$, $\therefore$ $a^2+b^2a^2=180$, and $a^2=\frac{180}{1+b^2}$; from (3), $2a^2+b^2a^2-3ba^2=0$, $\therefore$ $2+b^2-3b=0$, or $b^2-3b=-2$; hence, $b=2$, or 1, $\therefore$ $a^2=\frac{180}{1+b^2}=\frac{180}{5}=36$, $\therefore$ $a=6$. From (2), $c^2=144$, $\therefore$ $c=12$; $\therefore$ $\frac{2ac}{a+c}=8$, $\therefore$ &c. 20. Here $xy^3-x:xy^2-xy::37:12$, or $y^3-1:y(y-1)::37:12$, or $y^2+y+1:y::37:12$, $\therefore$ $12y^2-25y=-12$, $\therefore$ $y=\frac{4}{3}$. But $x+xy+xy^2+xy^3=700$, $\therefore$ $175x=18900$ and $x=108$, or 256. 21. Here $s=\frac{20}{2}[200+(20-1)100]=$ £21,000, and next, $s=\frac{20}{2}[400+19\times 200]=$ £42,000, $\therefore$ he is worth, in 40 years, £1,000 + £21,000 + £42,000 = £64,000. 22. Let $x=$ the No. of waggons attached, then $24-c\sqrt{x}=$ the speed of the train, c being a constant, to be determined; $\therefore$ by the question $20=24-c\sqrt{x}$; when $x=4$, $20=24-2c$; $\therefore$ $c=2$; hence $24-2\sqrt{x}=$ the speed of train with x waggons. If the speed is reduced to 0, $24-2\sqrt{x}=0$, $\therefore$ $\sqrt{x}=12$, and $x=144$; with one less it will move, $\therefore$ &c. 23. Let the digits be $x-y$, x, $x+y$, then $100(x-y)+10x+x+y$ expresses the number, therefore, by the question $\frac{111x-99y}{3x}=26$. Also, by the question $111x-99y+198=100(x+y)+10x+x-y$. These equations give $x=3$, $y=1$, $\therefore$ &c.

24. Let $x=$ No. of sides, then Euc. I., 32, cor 1, $120°+125°+130°+$ &c. $=2x\times 90°-4\times 90°$; and since there are as many angles as sides, the number of terms in the series is x; hence, $2x\times 90°-4\times 90°=180x-360=\frac{x}{2}[2a+(x-1)d]=\frac{x}{2}(240+5x-5)=\frac{x}{2}(235+5x)$; $\therefore 5x^2+235x-360x=-720$, or $x^2-25x=-144$; $\therefore$ &c.; the answer 16 applies when some of the angles are greater than 180°. 25. Here there is required the sum of the infinite series, whose first term is 20 and ratio $\frac{19}{20}$; $\therefore$ $s=\frac{a}{1-\frac{19}{20}}$ gives 400. 26. Here s, the sum

of the A.P., is the same in both cases, namely, $\frac{37}{2} = 18\frac{1}{2}$ leagues; then, since $s = \frac{n}{2}[2a + (n-1)d]$, transposing and dividing by n, we have $a = \frac{s}{n} - \frac{1}{2}(n-1)d$, and in the first case $n = 5$, $d = \frac{3}{2}$, $\therefore a = \frac{18\frac{1}{2}}{5} - \frac{3}{4}\cdot 4 = \frac{7}{10}$, the first day's march, &c. In the second case $n = 4$, $d = 2$; then $a = \frac{18\frac{1}{2}}{4} - (4-1)\times 1 = 1\frac{5}{8}$, the first day's march, &c. 27. Here is required the sum of the G. P., whose first term is $\frac{1}{4}$ and ratio 3; now the 20th power of 3 being 3486784401; taking 1 from this and dividing by $2(=3-1)$, we have the answer in farthings. 28. The space passed over will be twice the sum of the A. P., whose first term is 20 yds. = 60 ft., com. diff. 2 ft., and number of terms 200. Hence, $s = 100(120 + 199\times 2) = 51800$ ft., $\therefore$ the whole distance passed over $= 103600$ ft. = &c. 29. Let R and r be the radii, W and w the weights of such solid spheres, then $W : w :: R^3 : r^3$; $\therefore$ Art. 183., $W - w : W :: R^3 - r^3 : R^3$, $\therefore \frac{7}{8} : 1 :: R^3 - r^3 : R^3$, $\therefore 7 : 8 :: R^3 - r^3 : R^3$; hence Art. 183. $1 : 8 :: r^3 : R^3$, $\therefore 1 : 2 :: r : R$, which is the required ratio of the radii. 30. The value of a diamond weighing n carats $\propto n^2 = pn^2$; the value of a ruby weighing n carats $\propto n^{\frac{3}{2}} = qn^{\frac{3}{2}}$ where p and q are constants to be determined; then a diamond of a carats is worth pa^2, a ruby of b carats is worth $qb^{\frac{3}{2}}$; hence, according to the supposition $pa^2 = mqb^{\frac{3}{2}}\ldots(1)$, and $pa^2 + qb^{\frac{3}{2}} = c\ldots(2)$, in which p and q are to be found. Put in (2) the value of pa^2 from (1); then $mqb^{\frac{3}{2}} + qb^{\frac{3}{2}} = c \therefore q = \frac{c}{(m+1)b^{\frac{3}{2}}}$. Also from (1) and by multiplying (2) by m, $mpa^2 + pa^2 = mc$, $\therefore p = \frac{mc}{(m+1)a^2}$, $\therefore$ value of diamond $= pn^2 = \frac{mcn^2}{(m+1)a^2}$; of ruby $= qn^{\frac{3}{2}} = \frac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

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1. Here formula (1) Art. 222, n being 5, odd, gives $-1+15-90+270-405+243=32$. The same may be found as in Ex. p. 256. 2. Here $n=3$, $\therefore$ from same formula, $-1+15-45+35=4$; or thus

$$\begin{array}{ccccc} 1 & 5 & 15 & 35 & 70 \\ & 4 & 10 & 20 & 35 \\ & & 6 & 10 & 15 \\ & & & 4 & \end{array}$$

3. The series plainly is $3 \,.\, 1^3 + 5 \,.\, 2^3 + 7 \,.\, 3^3 + 9 \,.\, 4^3$, &c., or 3, 40, 189, 576, 1375, 2808, 5145, &c. The first terms of the orders of differences are 37, 112, 126, and $48=4$th term required; or to find 48 apply the formula as above, $n=4$ even; $3-160+1134-2304+1375=48$. We may illustrate the remarks in Art. 224 by means of this series. The two series constructed, as has been stated, will be

0 3 40 189 576 1375 2808 5145 8704, &c.,
3 37 149 387 799 1433 2337 3559, &c.

When $3=3$, $40=3+37$, $189=3+37+149$, $576=3+37+149+387$, $1375=3+37+149+387+799$, &c.; and the 9th term $8704=$ the sum of the first eight terms, and in general the $(n+1)$th term of the first series = the sum of n terms of the second, as is manifest from the relations of the numbers by the assumption. The differences to be substituted in (3) p. 258 for finding the sum of n terms are 3, 34, 78, and 48; those for use in (2) p. 257 are 37, 112, 126, and 48. To find the sum of 8 terms we have by (3) $s=8.a+\frac{8.7}{1.2}\cdot d_1+\frac{8.7.6}{1.2.3}\cdot d_2+\frac{8.7.6.5}{1.2\,3.4}d_3$; or $8\,.\,3+\frac{8.7}{1.2}\cdot 34+\frac{8.7.6}{1.2.3}\cdot 78+\frac{8.7.6.5}{1.2.3.4}\cdot 48=$ sum of 9 terms $=8704$. And by (2) to find the 9th term take 8th as above, $3+7\,.\,37+\frac{7.6}{1.2}\cdot 112+\frac{7.6.5}{1.2.3}\cdot 126+\frac{7.6.5.4}{1.2.3.4}\cdot 48=8704=$ the 9th term. 4. Apply here form. (2) p. 257, $n=20$, $a=3$, and d_1, d_2, &c., as in last Ex., there being only 4 terms as $d_5=0$ $\therefore$ 20th term $=3+703+19,\ 152+122,\ 094+186,\ 048=328000$. Also form. (3) Art. 224, $s=60+7030+127680+610470+744192=1489432$. 5. Here $a=2$, $d_1=4$, $d_2=2$, $d_3=0$; and the series will be found by (II) Art. 223; thus $a=2$, $b=6$, $c=2+8+2=12$, $d=2+12+6=20$, $e=2+$

$16+12+0=30, f=2+20+20+0=42, g=2+24+30+0=56$, $h=2+28+42+0=72$, &c. See coeffs. of binomial p. 92. Also by (2) p. 257 we have to put $n=6$, for 6th term $2+\frac{5}{1}\cdot 4+\frac{5.4}{1.2}\cdot 2=42$, the terms containing d_3, &c., vanishing. Also the 20th term is $2+\frac{19}{1}\cdot 4+\frac{19.18}{1.2}=2+76+342=420$.

6. Take here, $n=1, 2, 3, 4$, &c., successively, and we have $1.3^2+2.5^2+3.7^2+4.9^2+5.11^2+$ &c.

or	9 +	50 +	147 +	324 +	605 +	1014,	1575,	2701,	3610
$d_1=$		41	97	177	281	409	561	1026	
d_2			56	80	104	128			
d_3				24	24	24			

$\therefore d_1=41$, $d_2=56$, $d_3=24$, $d_4=0$; hence 100th term $=100(200+1)^2=4040100$; or from (2) p. 257, 100th term $=9+\frac{99}{1}\cdot 41+\frac{99.98}{1.2}\cdot 56+\frac{99.98.97}{1.2.3}24=4040100$.

7. Here $a=1$, $d_1=1$, $d_2=2$, $d_3=3$, $d_4=4$, $d_5=0$, and the series is found by substitution in (II) Art 223; then from (2) p. 257, and (3) p. 258, we have the two other answers. 8. Extend the series to 9 terms, 1, 3, 9, 27, 81, 243, 729, 2187, 6561; then the first terms of the successive orders of differences will be found to be, the first eight powers of 2, namely, 2, 4, 8, 16, 32, 64, 128, 256; or we may apply formula (1) Art. 222, in which $a=1$, $n=8$, b, c, d, &c., the successive terms of the given series. This gets $1-24+252-1512+5670-13608+20412-17496+6561=256$. 9. Here in the first series $d_1=2, d_2=1, d_3=0$ $\therefore$ formula (2) becomes $a+(n-1)2+\frac{(n-1)(n-2)}{1.2}\cdot 1=1+2n-2+\frac{n^2}{2}-\frac{3n}{2}+1=\frac{1}{2}n(n+1)$. In the 2nd series $d_1=4$, $d_2=2$, and the formula becomes $2+(n-1).4+\frac{(n-1)(n-2)\cdot 2}{1\cdot 2}=n(n+1)$, $\therefore$ the one is double the other. 10. The series plainly is $1^2.8+2^2.11+3^2.14+4^2.17+5^2.20$, &c., or 8, 44, 126, 272, 500, 828, 1274, &c., $\therefore d_1=36$, $d_2=46$, $d_3=18$, $d_4=0$; $\therefore$ from (3) p. 258, $s=5091500$. 11. Here the series being 1, 4, 10, 20, 35, &c., the differences are $d_1=3$, $d_2=3$, $d_3=1$, $d_4=0$; hence the sum to n terms is $s_n=n+$

$\frac{3n(n-1)}{1.2}+\frac{3n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}=\frac{3n^2}{2}\frac{n}{}+$ $\frac{n(n^2-3n+2)}{2}+\frac{n(n^3-6n^2+11n-6)}{2.3.4}=\frac{n(n^3+6n^2+11n+6)}{1.2.3.4}=$ $\frac{n}{1}\cdot\frac{n+1}{2}\cdot\frac{n+2}{3}\cdot\frac{n+3}{4}$. 12. Here $d_1=2, d_2=1, d_3=0, a=1, n=100$ $\therefore$ $s_{100}=100+9900+161700=171700$. Now (3) gives $s_n=n+\frac{n(n-1)}{1.2}\cdot 2+\frac{n(n-1)(n-2)}{1.2.3}\cdot 1=\frac{2n+2n^2-2n}{2}+$ $\frac{n(n^2-3n+2)}{1.2.3}=n^2+\frac{n(n^2-3n+2)}{6}=\frac{6n^2}{6}+\frac{n^3-3n^2+2n}{6}=$ $\frac{n^3+3n^2+2n}{6}=\frac{n(n^2+3n+2)}{6}=\frac{n(n+1)(n+2}{1.2.3}$; then, since $s_n=\frac{n(n+1)(n+2)}{2.3}$, $\therefore s_{100}=\frac{100.101.102}{6}=171700$.

13. Form. (3) p. 258, gives $d_1=2$, $d_2=0$, $s=60+\frac{60(60-1).2}{1.2}=3600$; and $s_n=n^2$. See Algebra, Ex. 5, p. 242, Ex. 12, p. 243, and Ex. 1, p. 258. 14. Assume for the sum of the series $1^2+2^2+3^2+\ .\ .\ .\ n^2$, $s=An+Bn^2+Cn^3$; then, by changing n into $n+1$, Art. 225, Ex. 1, we have $1^2+2^2+3^2+\ .\ .\ .\ .\ n^2+(n+1)^2=A(n+1)+B(n+1)^2+C(n+1)^3$; from this take $1^2+2^2+3^2+\ .\ .\ .\ n^2=An+Bn^2+Cn^3$ $\therefore$ $(n+1)^2=n^2+2n+1=A+2nB+B+3n^2C+3nC+C$. Then, equating the coefficients, we have the equations $3C=1$, $2B+3C=2$, $A+B+C=1$, which give $C=\frac{1}{3}$, $B=\frac{1}{2}$, $A=\frac{1}{6}$, $\therefore s=\frac{1}{6}n+\frac{1}{2}n^2+\frac{1}{3}n^3=\frac{n(n+1)(2n+1)}{1.2.3}$. Again, Art. 50, the general term of the 2nd series being $(2n-1)^2$. Assume $1^2+3^2+5^2+\ .\ .\ .\ .\ (2n-1)^2=An+Bn^2+Cn^3$; then, changing as before, n into $n+1$, and assuming a similar series, and taking the first from it, we have, since $2(n+1)-1=2n+1$, $(2n+1)^2=A+B(2n+1)+C(3n^2+3n+1)$, developing and equating in the usual way, we find $3C=4$, $3C+2B=4$, $C+B+A=1$ $\therefore C=\frac{4}{3}, B=0, A=-\frac{1}{3}$ $\therefore s=\frac{4}{3}n^3-\frac{1}{3}n=\frac{1}{3}n(4n^2-1)$. 15. Assume $1^4+2^4+3^4+\ .\ .\ .\ n^4=An+Bn^2+Cn^3+Dn^4+En^5$. Then change n into $n+1$, and subtract as before; then $n^4+4n^3+6n^2+4n+1=A+B(2n+1)+C(3n^2+3n+1)+D(4n^3+6n^2+4n+1)+E(5n^4+10n^3+10n^2+5n+1)$; and equating the coefficients of the like powers, we have the equations $5E=1$, $10E+4D$, $10E+6D+3C=6$, $5E+4$

$D+3C+2B=4$, $E+D+C+B+A=1$, which give $E=\frac{1}{5}$, $D=\frac{1}{2}$, $C=\frac{1}{3}$, $B=0$, $A=-\frac{1}{30}$ . . . ∴ &c. Again, assume $1^5+2^5+3^5+\ .\ .\ .\ n^5=An+Bn^2+Cn^3+Dn^4+En^5+Fn^6$, and proceed as above; we have $n^5+5n^4+10n^3+10n^2+5n+1=A+B(2n+1)+C(3n^2+3n+1)+D(4n^3+6n^2+4n+1)+E\times(5n^4+10n^3+10n^2+5n+1)+F(6n^5+15n^4+20n^3+15n^2+6n+1)$. Equating coefficients of like powers of n, we get $6F=1$; $15F+5E=5$; $20F+10E+4D=10$; $15F+10E+6D+3C=10$; $6F+5E+4D+3C+2B=5$; $F+E+D+C+B+A=1$. These equations give $F=\frac{1}{6}$; $E=\frac{1}{2}$; $D=\frac{5}{12}$; $C=0$; $B=-\frac{1}{12}$; $A=0$. Hence $1^5+2^5+3^5+\ .\ .\ .\ .\ n^5=\frac{1}{6}n^6+\frac{1}{2}n^5+\frac{5}{12}n^4-\frac{1}{12}n^2$. The sums of these series will be found by the differential method as follows:—

1	16	81	256	625	1296	2401
	15	65	175	369	671	1105
		50	110	194	302	434
			60	84	108	132
				24	24	24
					0	0

When $d_1=15$, $d_2=50$, $d_3=60$, $d_4=24$, $d_5=0$, and $a=1$ Then by (3) p. 258. $s=n+\frac{15n^2-15n}{2}+\frac{25n^3-75n^2+50n}{3}+\frac{5n^4-30n^3+55n^2-30n}{2}+\frac{n^5-10n^4+35n^3-50n^2+24n}{5}$; or

$s=\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$. Otherwise, in order to reduce the magnitude of the successive differences, we may take, instead of n terms of the above series, $n+1$ terms of the series 0, 1, 16, 81, 256, 625, 1296. The differences thus are—

1	15	65	175
	14	50	110
		36	60
			24

Whence in (3) p. 258, $a=0$, $d_1=1$, $d_2=14$, $d_3=36$, $d_4=24$, $d_5=0$, and $s=\frac{n^2+n}{1.2}\cdot+\frac{n^3-n}{1.2.3}\cdot 14+\frac{n^4-2n^3-n^2+2n}{1.2.3.4}\cdot 36+\frac{n^5-5n^4+5n^3+5n^2-6n}{1.2.3.4.5}\cdot 24$, which reduced, gives the same value as before. The sum of the other series would

be found in exactly the same way. 16. Assume, Art. 226, $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+$ &c., $=s$, omitting one factor; then, by transposition

$$\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\text{ \&c.,} = s-\frac{1}{3}$$

subtract, $$\frac{2}{15}+\frac{2}{35}+\frac{2}{63}, \text{ \&c.,} \quad = \frac{1}{3}$$

or, $$\frac{2}{3.5}+\frac{2}{5.7}+\frac{2}{7.9}+\text{ \&c.,} = \frac{1}{3}$$

$$\therefore \quad \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7\cdot 9}+\text{ \&c.,} = \frac{1}{6}$$

To find the sum to n terms, assume

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ \ldots\ \frac{1}{2n+1}+\frac{1}{2n+3}=s$$

transposing, $\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\ \ldots\ \frac{1}{2n+3}=s-\frac{1}{3}$; by subtraction,

$$\frac{2}{3.5}+\frac{2}{5.7}+\frac{2}{7.9}+\ \ldots\ \frac{2}{(2n+1)(2n+3)}+\frac{1}{2n+3}=\frac{1}{3}$$

$$\therefore \frac{2}{3.5}+\frac{2}{5.7}+\frac{2}{7.9}+\ \ldots\ +\frac{2}{(2n+1)(2n+3)}=\frac{1}{3}-\frac{1}{2n+3}=\frac{2n}{3(2n+3)}$$

and $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\therefore$ to n terms $=\frac{1.1}{3.2}-\frac{1.1}{2.(2n+3)}=\frac{n}{3(2n+3)}$

When n is infinite this is reduced to

$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ \ldots\ =\frac{1}{3.2}=\frac{1}{6}$, as before.

17. Omitting, as before, one factor from the denominator,

assume $$\frac{1}{2}+\frac{1}{5}+\frac{1}{8}+\text{ \&c. } = s.$$

$\therefore$ by transposing $\frac{1}{2}$ and subtracting,

$$\frac{1}{5}+\frac{1}{8}+\frac{1}{11}+\text{ \&c. } = s-\tfrac{1}{2}.$$

$$\therefore \quad \frac{3}{10}+\frac{3}{40}+\frac{3}{88}+\text{ \&c. } = \tfrac{1}{2}.$$

or, $$\frac{3}{2.5}+\frac{3}{5.8}+\frac{3}{8.11}+\frac{3}{11.14}\text{ \&c. } = \tfrac{1}{2},$$

Divide by 3; then $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\frac{1}{11.14}+$ &c. $=\frac{1}{6}$.

18. Assume $1.3^2 + 3.5^2 + 5.7^2 + \ . \ . \ . \ (2n-1)(2n+1)^2 = An + Bn^2 + Cn^3 + Dn^4$. Change n into $n+1$, and subtract; then $(2n+1)(2n+3)^2 = A + B(2n+1) + C(3n^2+3n+1) + D(4n^3+6n^2+4n+1)$; developing and equating, $4D = 8$, $6D + 3C = 28$, $4D + 3C + 2B = 30$, $D + C + B + A = 9$. These give $D = 2$, $C = \frac{16}{3}$, $B = 3$, $A = -\frac{4}{3}$; hence $s = 2n^4 + \frac{16}{3}n^3 + 3n^2 - \frac{4}{3}n = 2n^4 + 3n^2 + \frac{1}{3}(16n^3 - 4n) = n^2(2n^2+3) + \frac{1}{3}n(16n^2-4)$.

19. Assume $1.2. + 2.3 + 3.4 + \ . \ . \ . \ n(n+1) = An + Bn^2 + Cn^3$. Change n into $n+1$, and subtract; then $(n+1)(n+2) = A + B(2n+1) + C(3n^2+3n+1)$; developing and equating coeffs., $3C = 1, 3C + 2B = 3, C + B + A = 2, \therefore C = \frac{1}{3}, B = 1, A = \frac{2}{3}$, $\therefore \ s = \frac{2}{3}n + n^2 + \frac{1}{3}n^3 = \frac{1}{3}(2n + 3n^2 + n^3) = \frac{1}{3}n(n+1)(n+2)$. Again assume $1.2.3 + 2.3.4 + 3.4.5 + \ . \ . \ . \ + n(n+1)(n+2) = An^4 + Bn^3 + Cn^2 + Dn$. Change n into $n+1$ and subtract, then $(n+1)(n+2)(n+3) = n^3 + 6n^2 + 11n + 6 = A(4n^3 + 6n^2 + 4n + 1) + B(3n^2 + 3n + 1) + C(2n+1) + D$, $\therefore$ $4A = 1$; $6A + 3B = 6$; $4A + 3B + 2C = 11$; $A + B + C + D = 6$. Hence $A = \frac{1}{4}$; $B = \frac{3}{2}$; $C = \frac{11}{4}$; and $D = \frac{3}{2}$; $\therefore S = \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{11}{4}n^2 + \frac{3}{2}n = \frac{n}{4}(n^3 + 6n^2 + 11n + 6) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

20. Let $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14} + \&c = s \ldots (1)$ $\therefore$ $\frac{1}{5.8} + \frac{1}{8.11} + \frac{1}{11.14} + \&c. = s - \frac{1}{2.5} \ldots (2)$. From (1) take (2); then $\frac{1}{2.5} - \frac{1}{5.8} = \frac{6}{2.5.8}$; $\frac{1}{5.8} - \frac{1}{8.11} = \frac{6}{5.8.11}$ and $\frac{1}{8.11} - \frac{1}{11.14} = \frac{6}{8.11.14}$; hence $\frac{6}{2.5.8} + \frac{6}{5.8.11} + \frac{6}{8.11.14} + \&c. = \frac{1}{2.5}$, $\therefore$ $\frac{1}{2.5.8} + \frac{1}{5.8.11} + \frac{1}{8.11.14} + \&c. = \frac{1}{2.5.6} \ . \ . \ . \ (3)$ $\therefore$ $\frac{5}{2.5.8} + \frac{5}{5.8.11} + \frac{5}{8.11.14} + \&c. = \frac{5}{2.5.6} = \frac{1}{2.6} \ . \ . \ . \ (4)$ But Ex. 17., $\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \&c. = \frac{1}{2.3}$ and $\therefore \frac{8}{2.5.8} + \frac{11}{5.8.11} + \frac{14}{8.11.14} + \&c. = \frac{1}{2.3} \ldots (5)$. From (5) take (4); then $\frac{3}{2.5.8} + \frac{6}{5.8.11} + \frac{9}{8.11.14} + \&c. = \frac{1}{2.3} - \frac{1}{2.6} = \frac{1}{2.6}$. Take $\frac{2}{3}$ of this; then $\frac{2}{2.5.8} + \frac{4}{5.8.11} + \frac{6}{8.11.14} + \&c. = \frac{1}{2.6} \times \frac{2}{3} = \frac{2}{2.3.6} = \frac{1}{3.6} \ldots (6)$.

From (6) take (3); then $\frac{1}{2.5.8}+\frac{3}{5.8.11}+\frac{5}{8.11.14}+$ &c. $=$
$\frac{1}{3.6}-\frac{1}{2.5.6}=\frac{7}{2.3.5.6}$.

21. Assume here $\frac{1}{1.3.5}+\frac{1}{3.5.7}\cdots+\frac{1}{(2n-1)(2n+1)(2n+3)}$ $+\frac{1}{(2n+1)(2n+3)(2n+5)}=s$, the terms in n being those from which the successive denominators will be had on taking $n=1$; then transposing, $\frac{1}{3.5.7}+\frac{1}{5.7.9}+$ &c. $+$ $\frac{1}{(2n+1)(2n+3)(2n+5)}=s-\frac{1}{1.3.5}$. Hence by subtraction $\frac{6}{1.3.5.7}+\frac{6}{3.5.7.9}+\frac{6}{(2n-1)(2n+1)(2n+3)(2n+5)}+$ $\frac{1}{(2n+1)(2n+3)(2n+5)}=\frac{1}{1.3.5}$; transposing the last term on the right side, and incorporating it with $\frac{1}{1.3.5}$, and dividing by 6, we find $\frac{1}{1.3.5.7}+\frac{1}{3.5.7.9}+\frac{1}{5.7.9.11}+$ &c. to n terms $=$ $\frac{n(4n^2+18n+23)}{3.1.3.5(2n+1)(2n+3)(2n+5)}$. Now, as n increases towards infinity this tends to become $\frac{n.4n^2}{3.1.3.5.2n.2n.2n}$ or $\frac{1}{3.1.3.5.2}$ or $\frac{1}{1.3.5.6}$. Therefore the sum to infinity is $\frac{1}{3.5.6}=\frac{1}{90}$. In this Ex. the factors of the denominators are in A.P., and in such cases a series may be assumed having three factors in the denominators.

22. Take here $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. $=s\ldots(1)$

$$\therefore \qquad \frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+\text{ \&c. }=s-\frac{1}{2.3}\ldots(2).$$

subtract (2) from (1) $\frac{1}{3.4}+\frac{2}{3.4.5}+\frac{1}{3.4.5}$ &c. $=\frac{1}{2.3}$,

or $$\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}\text{ \&c. }=\frac{1}{2.3},$$

$$\therefore \qquad \frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+\text{ \&c. }=\frac{1}{3.4}=\frac{1}{12}.$$

23. Assume $\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\text{ \&c. }=s,$

$$\therefore \qquad \frac{1}{4.6}+\frac{1}{6.8}+\frac{1}{8.10}+\text{ \&c. }=s-\frac{1}{2.4},$$

subtract, $\frac{2}{4.6}+\frac{1}{6.8}+\frac{2}{4.6.10}+\text{ \&c. }=\frac{1}{2.4},$

or $\frac{4}{2.4.6}+\frac{4}{4.6.8}+\frac{4}{6.8.10}+\text{ \&c. }=\frac{1}{2.4},$

$$\therefore \qquad \frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\text{ \&c. }=\frac{1}{4.8}=\frac{1}{32}.$$

24. Assume $\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\text{ \&c. }=s,$

then, $\frac{1}{4.6.8}+\frac{1}{6.8.10}+\frac{1}{8.10.12}+\text{\&c. }=s-\frac{1}{2.4.6},$

subtract, $\therefore \frac{6}{2.4.6.8}+\frac{6}{4.6.8.10}+\frac{6}{6.8.10.12}+\text{ \&c. }=\frac{1}{2.4.6}=\frac{1}{48}$

$$\therefore \qquad \frac{1}{2.4.6.8}+\frac{1}{4.6.8.10}+\frac{1}{6.8.10.12}+\text{ \&c. }=\frac{1}{2.4.6.6}=\frac{1}{288}.$$

25. Applying here the method of Art. 226, Ex. 2, p. 264, assume

$$\frac{5}{1.2}+\frac{6}{2.3}+\frac{7}{3.4}+\frac{8}{4.5}+\text{\&c.}\;.\;.\;=s \quad \ldots \quad (1)$$

$\therefore$ by transposing, $\frac{6}{2.3}+\frac{7}{3.4}+\frac{8}{4.5}+\frac{9}{5.6}+\text{\&c.}\;.\;.\;=s-\frac{5}{1.2};$

or $\left(\frac{5}{2.3}+\frac{1}{2.3}\right)+\left(\frac{6}{3.4}+\frac{1}{3.4}\right)+\left(\frac{7}{4.5}+\frac{1}{4.5}\right)\;.\;.\;=s-\frac{5}{1.2};$

transpose, $\frac{5}{2.3}+\frac{6}{3.4}+\frac{7}{4.5}+\frac{8}{5.6} \qquad =s-\frac{5}{1.2}-$

$$\left(\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}\;.\;.\;.\right) \quad \ldots \quad (2)$$

Now the assumed series, (1), is the same as

$$\frac{15}{1.2.3}+\frac{24}{2.3.4}+\frac{35}{3.4.5}+\frac{48}{4.5.6}\;.\;.\;.=s \quad \ldots \quad (1')$$

subtract (2), $\frac{5}{1.2.3}+\frac{6}{3.4}+\frac{7}{4.5}+\frac{8}{5.6}\ \ldots = s-\frac{5}{1.2}-$

$$\left(\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}\ \ldots\right)$$

$$\therefore\quad \frac{10}{1.2.3}+\frac{12}{2.3.4}+\frac{14}{3.4.5}+\frac{16}{4.5.6}+\ \ldots\ \ldots = \frac{5}{1.2}+$$

$$\left(\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}\ \ldots\right)$$

Now the sum of the infinite series within the vinculum is easily found by these methods to be $\frac{1}{2}$; therefore dividing by 2 we have for the sum of the given series

$$\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}+\&c. = \frac{3}{2}.$$

26. Assume $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\ \ldots = s\ \ldots$ (1)

transpose, $\frac{2}{3.5.7}+\frac{3}{5.7.9}+\frac{4}{7.9.11}+\ \ldots = s-\frac{1}{1.3.5}$;

or, $\left(\frac{1}{3.5.7}+\frac{1}{3.5.7}\right)+\left(\frac{2}{5.7.9}+\frac{1}{5.7.9}\right)+\left(\frac{3}{7.9.11}+\frac{1}{7.9.11}\right)$

$$\ldots = s-\frac{1}{1.3.5}$$

transpose, $\frac{1}{3.5.7}+\frac{2}{5.7.9}+\frac{3}{7.9.11}+\ \ldots = s-\frac{1}{1.3.5}-$

$$\left(\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}\ \ldots\right)\ \ldots\ (2)$$

Now the assumed series (1) is the same as

$$\frac{7}{1.3.5.7}+\frac{18}{3.5.7.9}+\frac{33}{5.7.9.11}+\frac{52}{7.9.11.13}+\ldots = s\ldots(1')$$

subtract $\frac{1}{1.3.5.7}+\frac{2}{5.7.9}+\frac{3}{7.9.11}+\frac{4}{9.11.13}+\ldots$

$$= s-\frac{1}{1.3.5}-\left(\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}\ldots\right)$$

$$\therefore\quad \frac{6}{1.3.5.7}+\frac{12}{3.5.7.9}+\frac{18}{5.7.9.11}+\frac{24}{7.9.11.13}\ \ldots$$

$$= \frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}\ \ldots = \frac{1}{12}$$

$$\therefore\quad \frac{1}{1.3.5.7}+\frac{2}{3.5.7.9}+\frac{3}{5.7.9.11}+\frac{4}{7.9.11.13}+\ldots = \frac{1}{72}$$

That the sum of the series forming the right member is $\frac{1}{12}$ is easily shewn,

$$\text{Assume } \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.\ 9}+\ldots\ldots = s \quad \ldots \quad (1)$$

$$\therefore \quad \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\frac{1}{9.11}+\ldots\ldots = s-\frac{1}{1.3} \ldots (2)$$

$$\text{but series (1) equals } \frac{5}{1.3.5}+\frac{7}{3.5.7}+\frac{9}{5.7.9}+\frac{11}{7.9.11}+\ldots\ldots = s$$

$$\text{and series (2) ,, } \frac{1}{1.3.5}+\frac{3}{3.5.7}+\frac{5}{5.7.9}+\frac{7}{7.9.11}+\ldots = s-\frac{1}{1.3}$$

$$\text{subtract} \therefore \quad \frac{4}{1.3.5}+\frac{4}{3.5.7}+\frac{4}{5.7.9}+\frac{4}{7.9.11}+\ldots\ldots = \frac{1}{1.3}$$

$$\therefore \quad \frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\frac{1}{7.9.11}+\ldots = \frac{1}{12}$$

The following general formula, similar to that of Art. 227, is applicable to such series:—

$$\text{Since } \frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}= \frac{3pq}{n(n+p)(n+2p)(n+3p)} \therefore \frac{q}{n(n+p)(n+2p)(n+3p)} = \frac{1}{3p}\left\{\frac{q}{n(n+p)(n+2p)}-\frac{q}{(n+p)(n+2p)(n+3p)}\right\}.$$

In this Ex., 2 being the common difference of the terms of the denominator, $p=2$; q is successively 1, 2, 3 ... and the terms are formed by taking $n=1, 3, 5, \ldots$ &c.; then placing the positive terms given by the first fraction above, and the negative ones given by the second fraction below, the series takes the following form:—

$$s_\infty = \frac{1}{6}\left\{\begin{array}{l} \frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\text{\&c.} \\ \qquad\quad -\frac{1}{3.5.7}-\frac{2}{5.7.9}-\text{\&c.} \end{array}\right\}$$

$$=\frac{1}{6}\left\{\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\text{\&c.}=\frac{1}{12}\right\}$$

Then $s=\frac{1}{72}$ as before.

27. Assume $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots = s.$

$$\therefore \quad \frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots=s-1+\frac{1}{3}=s-\frac{2}{3}$$

subtract, $\frac{4}{1.5}-\frac{4}{3.7}+\frac{4}{5.9}-\frac{4}{7.11}+\ldots=\frac{2}{3}$

$$\therefore \quad \frac{1}{1.5}-\frac{1}{3.7}+\frac{1}{5.9}-\frac{1}{7.11}+\ldots=\frac{1}{6}$$

28. Assume $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots=s$

transpose, $-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\ldots=s-1+\frac{1}{2}-\frac{1}{3}$

subtract $\frac{5}{1.4}-\frac{7}{2.5}+\frac{9}{3.6}-\frac{11}{4.7}+\ldots=1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

Otherwise, the series may be put under the following forms:—

$$\left(\frac{1}{1}+\frac{1}{4}\right)-\left(\frac{1}{2}+\frac{1}{5}\right)+\left(\frac{1}{3}+\frac{1}{6}\right)-\left(\frac{1}{4}+\frac{1}{7}\right)+\ldots=$$
$$\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ \ldots\right)+\left(\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\ldots\right)$$

where the first terms within the vincula are collected into one set, and those in the second place into another; and after the first three, the terms are the same but with contrary signs; hence the sum is $\frac{1}{1}-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$.

29. Assume $\frac{1}{a}+\frac{1}{a+x}+\frac{1}{a+2x}+\frac{1}{a+3x}+\ \cdots\ =s$

$$\therefore \quad \frac{1}{a+x}+\frac{1}{a+2x}+\frac{1}{a+3x}+\frac{1}{a+4x}+\ \cdots\ =s-\frac{1}{a}$$

subtract, $\frac{x}{a(a+x)}+\frac{x}{(a+x)(a+2x}+\frac{x}{(a+2x)(a+3x)}+\ \ldots\ =\frac{1}{a}$

$$\therefore \quad \frac{1}{a(a+x)}+\frac{1}{(a+x)(a+2x)}+\frac{1}{(a+2x)(a+3x)}+\ \cdots\ =\frac{1}{ax}$$

30. Put $a+2ar+3ar^2+4ar^3+\ldots+nar^{n-1}=s$. Multiply by r, $ar+2ar^2+3ar^3+4ar^4+\ .\ .\ .\ +(n-1)ar^{n-1}+nar^n=rs$. Subtract, $a+ar+ar^2+ar^3+\ldots\ ar^{n-1}-nar^n=s-sr$;

or Art. 209 (1), $\frac{a-ar^n}{1-\ r}-nar^n=s\,(1-r)$.

$$\therefore s=\frac{a(1-r^n)}{(1-r)^2}-\frac{nar^n}{1-r}$$

When n is infinite there is only one real value corresponding to $r<1$, namely, $\frac{a}{(1-r)^2}$; $r=1$, $r>1$ give infinite values.

31. Here $5x^2 = ax \,.\, 3x + a'x^2 \,.\, 1 = (3a + a')x^2$

also $7x^3 = ax \,.\, 5x^2 + a'x^2 \,.\, 3x = (5a + 3a')x^3$

$$\therefore \quad \begin{matrix} 3a + a' = 5, \\ 5a + 3a' = 7, \end{matrix} \therefore \begin{cases} a = 2 \\ a = -1 \end{cases}; \text{ scale, } 2x, -x^2$$

or, equating the corresponding co-efficients, we might put at once $3a + a' = 5$, $5a + 3a' = 7$. Also, $A = 1$, $A_1 = 3$. Substitute these values in formula (2), p. 272; $\therefore$ &c. The sums of recurring series may all be verified by actual division, Arts. 81, 83.

32. Here $7x^2 = ax \,.\, 2x + a'x^2 \,.\, 1 = (2a + a')x^2$, and $20x^3 = ax \,.\, 7x^2 + a'x^2 \,.\, 2x = (7a + 2a')x^3$; $\therefore$ $2a + a' = 7$, $7a + 2a' = 20$; $\therefore$ $a = 2$, $a' = 3$, and the scale is $2x$, $3x^2$; also $A = 1$, $A_1 = 2$; $\therefore$ &c. 33. Here $3x^2 = ax \,.\, 2x + a'x^2 \,.\, 1 = (2a + a')x^2$. $4x^3 = ax \,.\, 3x^2 + a'x^2 \,.\, 2x = (3a + 2a')x^3$ $\therefore$ $2a + a' = 3$, $3a + 2a' = 4$; $\therefore$ $a = 2$, $a' = -1$; scale $2x, -x^2$. Also $A = 1$, $A_1 = 2$; $\therefore$ &c.

34. Here $-2a + a' = 3$, $3a - 2a' = -4$; $\therefore$ $a = -2$, $a' = -1$; scale, $-2x, -x^2$; $A = 1, A_1 = -2$; $\therefore$ &c. 35. The equations are $21x^2 = ax \,.\, 9x + a'x^2 \,.\, 4 = (9a + 4a')x^2$; $51x^3 = ax \,.\, 21x^2 + a'x^2 \times 9x = (21a + 9a')x^3$; $\therefore$ $9a + 4a' = 21$, $21a + 9a' = 51$; $\therefore a = 5, a' = -6$; scale $5x, -6x^2$; also $A = 4$, $A_1 = 9$; $\therefore$ 36. Put $6a + a' = 12$, $12a + 6a' = 48$; $\therefore a = 1, a' = 6$, scale $x^2, +6x$; $A = 1$, $A_1 = 6$, &c. 37. Here $2a + a' = 3$, $3a + 2a' = 4$; $\therefore$ $a = 2$, $a' = -1$; $\therefore$ scale is $2x, -x^2$; and $A = 1$, $A_1 = 2$, &c.

38. Here Art. 233, p. 272, note, we assume a, a', a'' for the scale, then

$$\begin{aligned} 49x^3 &= ax \,.\, 25x^2 + a'x^2 \,.\, 9x + a''x^3 \,.\, 1; \quad \therefore 25a + 9a' + a'' = 49 \\ 81x^4 &= ax \,.\, 49x^3 + a'x^2 \,.\, 25x^2 + a''x^3 \,.\, 9x; \quad \therefore 49a + 25a' + 9a'' = 81 \\ 121x^5 &= ax \,.\, 81x^4 + a'x^2 \,.\, 49x^3 + a''x^3 \,.\, 25x^2; \quad \therefore 81a + 49a' + 25a'' = 121 \end{aligned}$$

These three equations give $a = 3$, $a' = -3$, $a'' = 1$; $\therefore$ the scale is $3x, -3x^2, +x^3$; also $A = 1$, $A_1 = 9$, $A_2 = 25$. These values are to be substituted in formula (3) which then becomes

$$s = \frac{1 + 9x + 25x^2 - 3x - 3x \,.\, 9x + 3x^2}{1 - 3x + 3x^2 - x^3} = \frac{1 + 6x + x^2}{(1-x)^3}$$

39. The equations in this case are—

$$\begin{aligned} 9a + 4a' + a'' &= 16 \\ 16a + 9a' + 4a'' &= 25 \\ 25a + 16a' + 9a'' &= 36 \end{aligned}$$

These give $a=3$, $a'=-3$, $a''=1$; $\therefore$ scale is $3x-3x^2+x^3$; and $A=1$, $A_1=4$, $A_2=9$; then substitute these in (3).

40. Here $3a+a'=5$, $5a+3a'=7$; $\therefore$ $a=2$, $a'=-1$; $\therefore$ scale is $2.\frac{1}{x}-1\frac{1}{x^2}$; also $A=1$, $A_1=3$; substitute these values in (2), then

$$s=\frac{1+\frac{3}{x}-\frac{2}{x}}{1-\frac{2}{x}+\frac{1}{x^2}}=\frac{1+\frac{1}{x}}{\frac{(x-1)^2}{x^2}}=\frac{x+1}{x}\div\frac{(x-1)^2}{x^2}=\frac{x(x+1)}{(x-1)^2}.$$

If the even terms of the series are negative, the scale is, -2, -1; and $s=\dfrac{x(x-1)}{(x+1)^2}$.

41. Here, note p. 273, we may find the scale by taking the nth order of differences; we thus get

1st order	7	19	37	61	91	127
2nd „		12	18	24	30	36
3rd „			6	6	6	6
4th „						0

Here $\therefore n=4$; as the 4th order vanishes; then substituting n in the formula in the note we find for the scale of relation $4, -6, 4, -1$, or $4x$, $-6x^2$, $+4x^3-x^4$; the denominator will thus consist of five terms, of which 1 is the first; and as all the terms in this denominator, except the first, are negative, and two terms in the scale are negative, the denominator will be $1-4x+6x^2-4x^3+x^4=(1-x)^4$; the numerator will be determined from the numerator of form. (3), p. 272. As the terms involving x^3 vanish, it was thought unnecessary to give in the Algebra the extension of form. (3) applicable to a scale of four terms; but it may be given here. It is the following:—

$$\frac{A+A_1x+A_2x^2+A_3x^3-ax(A+A_1x+A_2x^2)-a'x^2(A+A_1x)-a''x^3A}{1-ax-a'x^2-a''x^3-a'''x^4}.$$

stituting in this the numbers of the Ex., we have

$$\frac{1+8x+27x^2+64x^3-4x-4x.8x-4x.27x^2+6x^2+48x^3-4x^3}{(1-x)^4}=\frac{1+4x+x^2}{(1-x)^4}$$

the other terms of the numerator cancelling.

42. The scale is most readily found here as in the last

exercise by taking the differences and using the formula in the note, p. 273, for $n=3$; the equations are:—

$$\begin{aligned} 35a+15a'+\ \ 3a'' &= \ \ 63 \\ 63a+35a'+15a'' &= \ \ 99 \\ 99a+63a'+35a'' &= 143; \end{aligned}$$

in either way we get $a=3$, $a'=-3$, $a''=1$, and the scale is $3x-3x^2+x^3$; also $A=3$, $A_1=15$, $A_2=35$; substituting these values in (3) we find

$$s=\frac{1+6x-x^2}{(1-x)^3}.$$

43. Here, formula (2), Art. 229, $\dfrac{n(n+1)(n+2)}{1.2.3}=4960.$

44. By formula (1), Art. 228, $\dfrac{n(n+1)(2n+1)}{1.2.3}=9455.$

45. Here, formula (3), Art. 230, $\dfrac{n(n+1)(3l-n+1)}{1.2.3}=23405.$

. . . . 47. In triangular pile 1540. In the complete rectangular pile, the number as in Ex. 45 is 7070; in the incomplete pile $20-12=8$, and $40-12=28$. $\therefore$ Number of balls $=\dfrac{8.9.(3.28-8+1)}{1.2.3}=4.3.77=924$; then $7070-924=6146=$ the number in the incomplete pile.

48. Here the square pile, if complete, would contain $\dfrac{24.25.49}{1.2.3}=4.25.49=4900$ balls; and as there are 8 balls in the side of the top of the truncated pile, and (Art. 228) the number of courses is the same as the number of balls in one side of the lowest, 7 courses must be superimposed to complete the pile; hence the wanting part would contain $\dfrac{7.8.15}{1.2.3}=7.4.5=140$ balls; $\therefore$ $4900-140=4760$, is the number in the incomplete pile.

49. The number of balls in the nth layer of a triangular pile is $\dfrac{n(n+1)}{2}$, and in the $(n-1)$th layer $\dfrac{n(n-1)}{2}$, Art. 229, $\therefore$ the sum of these is the number in the two lowest layers; but $\dfrac{n(n+1)}{2}+\dfrac{n(n-1)}{2}=n^2$; and n^2 is the number in the base of a square pile of n layers, $\therefore$ &c.

50. It is manifest, Arts. 229, 228, that we have here

$\frac{n(n+1)(n+2)}{1.2.3} : \frac{n(n+1)(2n+1)}{1.2.3} = 6 : 11$; that is, $n+2 : 2n+1 = 6 : 11$ whence $n = 16$, and the numbers are 816 in the triangular and 1496 in the square pile.

XXIV. LOGARITHMS AND EXPONENTIALS.

p. 289.

1. Here Art. 241, x log. a = log. b, ∴ &c. 2. bx log. a = log. c, ∴ &c. 3. x log. $20 = 2$ ∴ $x = \frac{2}{1{\cdot}30103}$, &c. 4. Here $25 = \frac{100}{4} = \frac{100}{2^2}$; ∴ log. $25 = 2 - 2$ log. 2; ∴ &c., and $\cdot 0125 = \frac{1}{80}$ = log. 1 − (log. 10 + 3 log. 2) = − (log. 10 + 3 log. 2) = $-1{\cdot}903090 = -2 + {\cdot}096910$. 5. Here x log. $\frac{5}{4}$ = log. $\frac{109}{2}$ = log. 109 − log. 2.

∴ x(log. 5 − log. 4) = log. 109 − log. 2.

or $x(\cdot 096910) = 1{\cdot}736397$ ∴ &c.

6. Here mx log. $a + nx$ log. b = log. c ∴ $x(m$ log. $a + n$ log. $b)$ = log. c; ∴ $x = \frac{\log. c}{m \log. a + n \log. b}$.

7. Here log. 4 = 2 log. 2 = ·602060; log. 6 = log. 2 + log. 3 = ·778151; log. 8 = 3 log. 2 = ·903090; log. 9 = 2 log. 3 = ·954242; log. 10 = 1 log. 21 = log. 3 + log. 7 = 1·322219; log. 32 = 5 log. 2 = 1·505150; log. 20 = log. 10 + log. 2 = 1·301030; log. 50 = log. 10 + log. 5 = 1 + log. 5 = 1·698970; log. 40 = log. 10 + 2 log. 2 = 1·602060.

8. Here log. $11 = 1 + {\cdot}86858896\left(\frac{1}{21} + \frac{1}{3{\cdot}21^3} + \frac{1}{5}\frac{1}{21^5} + \text{\&c.}\right)$

$= 1 + {\cdot}86858896\ ({\cdot}0476555\ .\ .)$

$= 1{\cdot}041393.$

Similarly log. 12 = 1·079181; and log. 13 = 1·113943; ∴ log. 33 = log. 11 + log. 3 = 1·518514; log. 39 = log. 13 + log. 3 = 1·591064; log. 121 = 2 log. 11 = 2·082786; log. 143 = log. 11 + log. 13 = 2·155336; log. 169 = 2 log. 13 = 2·227886.

9. Taking the logs. of both side, we have

$3x$ log. 2 + $(2x - 1)$ log. 5 = $5x$ log. 4 + $(x + 1)$ log. 3.

= $10x$ log. 2 + $(x + 1)$ log. 3.

∴ x(2 log. 5 − 7 log. 2 − log. 3) = log. 3 + log. 5.

$$\therefore\ x = \frac{1{\cdot}176091}{-1{\cdot}186391} = -{\cdot}991.$$

The denominator $-1{\cdot}186391$ is usually written, for the sake of uniformity in the notation (Art. 239, note), $\bar{2}{\cdot}813609$, that is, $\bar{2} + {\cdot}813609$.

10. Here $ar^{n-1} = 64$; $\therefore$ log. $a + (n-1)$ log. $r =$ log. 64. Whence, since $a = 729$, $r = \frac{2}{3}$

$$\therefore \qquad n = \frac{1{\cdot}232639}{{\cdot}176091} = 7.$$

Also from the formula under Ex. 6, p. 289, we have

$$\log.\ \{a - s(1-r)\} - \log.\ a = n \log.\ r,$$

that is, $$-1{\cdot}584820 = n(-{\cdot}176091),$$

$$\therefore \qquad n = \frac{1{\cdot}584820}{{\cdot}176091} = 9.$$

11. Here y log. $x = x$ log. y; and p log. $x = q$ log. y, $\frac{y}{x} = \frac{p}{q}$; also $\frac{p}{q}$ log. $x =$ log. y, hence log. $y -$ log. $x = \frac{p}{q}$ log. x; and $\frac{p}{q}$ to x - log. $x =$ log. $\frac{p}{q}$, $\therefore$ $\frac{p \log.\ x - q \log.\ x}{q} = \frac{p-q}{q}$ log. $x =$ log. $\frac{p}{q}$, $\therefore$ log. $x = \frac{q}{p-q}$ log. $\frac{p}{q}$. But this (Art. 241, par. 2nd) is the same as log. $\left(\frac{p}{q}\right)^{\frac{p}{p-q}}$, $\therefore$ log. $x =$ log. $\left(\frac{p}{q}\right)^{\frac{q}{p-q}}$, $\therefore$ $x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}$. But $y = \frac{p}{q}x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}+1}$ $= \left(\frac{p}{q}\right)^{\frac{p}{p-q}}$. Or thus without the use of logarithms: since $x^y = y^x$, $x^{\frac{y}{x}} = y = x^{\frac{p}{q}}$, $\therefore$ $\frac{y}{x} = \frac{p}{q}$; $\therefore$ $\frac{x^{\frac{p}{q}}}{x} = \frac{p}{q}$, $\therefore x^{\frac{p}{q}-1}$ or $x^{\frac{p-q}{q}} = \frac{p}{q}$; hence evolving, $x = \left(\frac{p}{q}\right)^{\frac{q}{p-q}}$, and as before $y = \frac{p}{q}x = \left(\frac{p}{q}\right)^{\frac{p}{p-q}}$.

12. Here log. $5 =$ log. $\frac{10}{2} = 1 -$ log. $2 = {\cdot}698970$; log. $16 = 4$ log. $2 = 1{\cdot}204120$; log. ${\cdot}016 =$ log. $\frac{16}{10^3} =$ log. $16 - 3$ log. 10 $=$ log. $16 - 3 = \bar{2}{\cdot}204120$ (Art. 239 and note); log. $6{\cdot}25 =$

log. $\frac{625}{10^2} = 4$ log. $5 - 2$ log. $10 = 4$ log. $5 - 2 = \cdot 795880$; log. $3\frac{1}{8} =$ log. $\frac{25}{8} = 2$ log. $5 - 3$ log. $2 = \cdot 494850$.

13. Here, Art. 235, if we suppose the 4th differences to vanish, we shall have, form. (I), Art. 223, $d_4 = 0 = e - 4d + 6c - 4b + a$; and $\therefore$ log. $84 - 4$ log. $83 + 6$ log. $82 - 4$ log. $81 +$ log. $80 = 0$. Hence log. $83 = \frac{\text{log. } 84 + 6 \text{ log. } 82 + \text{log. } 80}{4} -$ log. 81, and by substituting the given values of the logs. we find log. $83 = 1\cdot 919078175$. The log. of the Tables being $1\cdot 9190782$. And hence the assumption that the 4th differences vanish is near the truth.

14. Supposing, as above, the 4th differences to vanish, we shall have the same expression as before; we may here put it under the form

$$c = \frac{4(b+d) - (a+e)}{6} = \frac{16\cdot 1025340 - 4\cdot 0255107}{6}$$

$\therefore$ log. $103 = 2\cdot 0128372$.

XXV. PERMUTATIONS AND COMBINATIONS.

p. 296.

1. Here $n = 12$; apply form. (1), (2), (3), p. 291, and also in Ex. 2, 3.

. . . 4. Apply form. (6) to the word Algebra when $n = 7$, $p = 2$; $\therefore$ $P = \frac{7.6 \ . \ . \ . \ 1}{1. \ 2.}$; form. (7), (8) apply to the remaining words.

5. Apply form. (8); the denominator is 1.2.3.1.2.3.4.5.1.2.

6. Apply form. (9), Art. 257; in this case $\frac{n(n-1)}{1.2} = \frac{24.23}{1.2}$; $\frac{n(n-1)(n-2)}{1.2.3} = \frac{24.23.22}{1.2.3}$; $\frac{n(n-1)(n-2(n-3)}{1.2.3.4}$ &c.

7. The combinations are here complementary to those in Ex. 6; since $22 + 2 = 24$, &c. See the Example, top of p. 294.

8. Here 11 and 13 being complementary, making up 24,

the number of combinations is the same in both cases; with 12, even, the number is greatest. See Art. 259.

9. The series is 9, 36, 84, 126, 126, 84, 36, 9; the greatest combinations are $\frac{n-1}{2}$, or $\frac{n+1}{2}$, together, by Art. 259; *i.e.*, 4, or $\frac{n-1}{2}$, and 5, or $\frac{n+1}{1}$, alike give 126.

10. Here $n-r+1=40$; $\therefore \frac{52.51.50\ldots.40}{1.2.3\ldots..13}$ gives the No.

11. The same in both cases, as $10=3+7$.

12. Here $n(n-1)(n-2).1=6.\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}$; $\therefore$ $1=\frac{n-3}{4}$, and $n=7$.

13. Thatis, combinations of 25 things taken 4 together, then $c_4=\frac{25.24.23.22}{1.2.3.4}$, for $n-r+1=22$. Again, $25-1=24$ men remain; and any one man can be joined to as many combinations of 3 as can be formed out of the 24 men who remain; $\therefore \frac{24.23.22}{1.2.3}$ is the number of nights.

14. Here $n=20$, and $r=6\times 3=18$; then there are the two combinations $\frac{20.19}{1.2}$; and $\frac{19.18}{1.2}$, as in Ex. 13.

15. Here 6.5.4.3.2.1 expresses the number of possible arrangements. But the question may be viewed differently; we may suppose one person to remain fixed and the others arranged, so that no one shall have the same neighbours in the same position; this would give $5.4.3.2.1=120$; but if the arrangement B, A, C, be regarded as the same as C, A, B, the position of neighbours, right and left, *not* being regarded as making a difference, then one-half of the arrangements will be similar to the other, and the number will be $\frac{1}{2}$. $120=60$.

16. Here, by Art. 261, the combinations will be found by taking the product of two expressions similar to form. (9), there being two sets of things; and as $n=12$, $r=6$, $n-r+1=7$; and $n-r+1=3$; $\therefore \frac{12.11.10\ldots.7}{1.2.3\ldots.6}\times\frac{5.4.3}{1.2.3}$ is the number.

17. Here, as in Ex. 16, the expression is $\frac{12.11.10}{1.2.3} \times \frac{16 \ldots . 13}{1.2 \ldots 4}$.

18. There are in the polygon n angular points, by connecting any three of which a triangle will be formed; the number of triangles will therefore be the number of combinations of n things taken three together, that is

$$\frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}.$$

19. Here, Art. 260, the number of combinations is had by taking the continual product, 10.8.7.5; also, on the second supposition, the number is $p^4 = 10{,}000$.

20. The number is greatest (Art. 259) when the value of r is $\frac{n}{2}$, in this case 4; $\therefore c_4 = \frac{8.7.6\,5}{1.2.3.4} = 70$.

21. The 6 coins may be combined any number together to make a sum; if taken 6 together, the number of different sums is expressed by $\frac{6.5.4.3.2.1}{1.2.3.4\,5.6} = 1$, (9); if taken 5 together, the last term, $n - r + 1 = 2$, and the number of different sums is $\frac{6.5.4.3.2}{1.2.3.4.5} = 6$; if 4 together, $\frac{6.5.4.3}{1.2.3.4} = 15$, &c.; also if taken 1 together, the number is the same as if taken 5 together, 5 and 1 being $n - r$ and r, that is these combinations are complementary, Art. 257. Hence the number of different sums is expressed by

$$1 + 6 + \frac{6.5}{1.2} + \frac{6.5.4}{1.2.3} + \frac{6.5}{1.2} + 6 = 63.$$

in general the sum is $2^n - 1$.

22. There are 7 consonants and 3 vowels in the first 10 letters of the alphabet; then because of the 6 letters in each word 2 are vowels, these 4 combinations of 7 consonants, or 7 taken 4 together, that is, $\frac{7.6.5.4}{1.2.3.4} = 35$; also the number of combinations of 3 vowels taken 2 together is $\frac{3.2}{1.2} = 3 \therefore$ $35.3 = 105$ is the number of different sets of words having 4 consonants and 2 vowels; but the number of permutations of the 6 letters taken all together is $6.5.4.3.2.1 = 720$; $\therefore$ the whole number of words that can be formed is $105.720 = 75{,}600$.

23. Let $n =$ the number of cards, then by the question $\frac{n(n-1)(n-2)}{1.2.3} = 425n$; dividing by n and clearing, $n^2 - 3n + 2 = 2550$; a quadratic readily giving 52 for n.

24. A man may vote for 1 only, for any 2, or any 3; now the number of ways of voting for 1 is 4; of voting for 2 $\frac{4.3}{1.2} = 6$, *i.e.*, 4 things 2 together, of voting for 3, $\frac{4.3.2}{1.2.3} = 4$; $\therefore$ total number $= 4 + 6 + 4 = 14$ ways.

XXVI. BINOMIAL THEOREM, p. 308.

1. Here $\left(1+\frac{x}{2}\right)^5 = 1 + 5.\frac{x}{2} + \frac{5.4}{1.2}.\left(\frac{x}{2}\right)^2 + \frac{5.4.3}{1.2.3}.\left(\frac{x}{2}\right)^3 + \frac{5.4.3.2}{1.2.3.4}.\left(\frac{x}{2}\right)^4 + \frac{5.4.3.2.1}{1.2.3.4.5}.\left(\frac{x}{2}\right)^5 =$ &c.; $\left(1-\frac{x}{2}\right)^{-2} = 1 + 2.\frac{x}{2} + \frac{2.3}{1.2}\left(\frac{x}{2}\right)^2 + \frac{2.3.4}{1.2.3}\left(\frac{x}{2}\right)^3 + \frac{2.3.4.5}{1.2.3.4}.\left(\frac{x}{2}\right)^4 =$ &c.; $(a+2b-c)^3 = \{a+(2b-c)\}^3 = a^3 + 3a^2(2b-c) + \frac{3.2}{1.2}a(2b-c)^2 + (2b-c)^3 = a^3 + 6a^2b - 3a^2c + 3a(4b^2 - 4bc + c^2) + 8b^3 - 12b^2c + 6bc^2 - c^3 =$ &c. Also $\left(\frac{a+x}{a-x}\right)^n = \left(1+\frac{2x}{a-x}\right)^n = 1 + n\left(\frac{2x}{a-x}\right) + \frac{n(n-1)}{1.2.} \times \left(\frac{2x}{a-x}\right)^2 + \frac{n(n-1)(n-2)}{1.2.3}.\left(\frac{2x}{a-x}\right)^3 +$ &c.

2. Apply the formula and example of Art. 271; r is in this case the first whole number greater than $(\frac{3}{2}+1)\frac{5}{2}.\frac{1}{1+\frac{5}{2}} = 1\frac{11}{14}$, that is, 2; in the 2nd case the general factor $-\frac{n+r-1}{r} \times \frac{x}{a} = -\frac{\frac{3}{2}+r-1}{r} \times \frac{5}{2} = -\frac{r+\frac{1}{2}}{r} \times \frac{5}{2}$ is always greater than 1; this shews that the series increases continually; in the third case r is next term $> 1\frac{9}{11}$, that is the second.

3. Here $(\frac{3}{2}+1).\frac{9}{10} \div (1+\frac{9}{10}) = \frac{90}{76} = 1\frac{7}{38}$; $\therefore r = 2$, and the series begins to converge with the 3rd term. Or thus, the series will begin to converge at the $(r+1)$th term. When

$$\frac{\frac{3}{2}-r+1}{r}.\frac{9}{10} < 1, \text{ that is } \frac{5-2r}{2r}.\frac{9}{10} < 1.$$

Or, $20r > 45 - 18r$, that is $38r > 45$ or $r > 1\frac{7}{38}$.

4. There are $n+1=13$ terms, $\therefore$ the 7th is the middle term; and 7th term $=\frac{12.11\ldots.7}{1.\ 2\ldots.6}a^{12}x^{12}=924a^{12}x^{12}$; also the two middle terms are the 7th and 8th for $n+1=14$; 7th term $=$ $\frac{13.12\ldots8}{1.\ 2\ldots6}=1716a^7x^6$; 8th $=\frac{13.12\ldots.7}{1.\ 2\ldots.7}=1716a^6x^7$.

5. The rth term (Art. 269) will be

$$\frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)\ldots.(\frac{1}{3}-r+2)}{1.2.3\quad\ldots.\quad(r-1)}.a^{\frac{1}{3}-r+1}x^{r-1}$$

$$=\frac{\frac{(1)(-2)(-5)\ldots(-3r+7)}{3^{r-1}}}{1.2.3.\quad\ldots\quad(r-1)}a^{\frac{4-3r}{3}}.x^{r-1}$$

$$=\frac{(1)(-2)(-5)\ldots(-3r+7)}{1.2.3\quad\ldots\quad(r-1)}.a^{\frac{4-3r}{3}}.\frac{x^{r-1}}{3^{r-1}}.$$

Then to make the terms of the numerator positive, multiply by $(-1)^{r-2}$, $(r-2)$ being the number of negative factors; we thus get

$$r\text{th term }=(-1)^{r-2}\frac{1.2.5\ldots(3r-7)}{1.2.3\ldots(r-1)}a^{\frac{4-3r}{3}}.\left(\frac{x}{3}\right)^{r-1}.$$

Again the 7th term of $(a^3+3ax)^9$ is $\frac{9.8\ldots.4}{1.2\ldots.6}(a^3)^3(3ax)^6=$ $61236a^{15}x^6$.

6. The number of terms is here $2n+1$, odd, since the index is even; the middle term will be an odd term with n terms on each side; the middle term is thus $n+1$; then applying the formula, introducing the factor $1.2.3\ldots.n$, and arranging the terms, we have mid. term $=(n+1)$th

$$=\frac{2n(2n-1)\,(2n-2)\ldots\ldots(n+1)}{1\,.\,2\,.\,3\quad\ldots\quad n}.x^n$$

$$=\frac{1.2.3\ldots n(n+1)\ldots(2n-1)2n}{(1.2.3\ldots n)^2}.x^n$$

$$=\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}\cdot\frac{2.4.6\ldots2n}{1.2.3\ldots n}.x^n$$

$$=\frac{1.3.5\ldots\ldots.(2n-1)}{1.2.3\ldots n}2^nx^n$$

Again, the coeff. of . th term of $(1+x)^n=\frac{n(n-1)\ldots(n-r+2)}{1.2\,.\,3\ldots(r-1)}$;

coeff. of $(r+1)$th term $= \dfrac{n(n-1)\ldots.(n-r+2)}{1.2.\ldots(r-1)} \times \dfrac{n-(r-1)}{r}$,
(Art. 270.)

$$= \left\{1+\frac{n-(r-1)}{r}\right\} \cdot \frac{n(n-1)\ldots(n-r+2)}{1.2..(r-1)}$$

$$= \frac{(n+1)\,n\,(n-1)\ldots\{n-(r-2)\}}{1.2.3\ldots r}$$

or, $$\frac{(n+1)\,n\,(n-1)\ldots\{(n+1)-(r-1)\}}{1.2.3\ \ldots\ r}$$

which is the co-efficient of the $(r+1)$th term of $(1+x)^{n+1}$.

XXVII. PROBABILITY, p. 317.

1. The probability of drawing a white ball first is $\frac{6}{11}$; and if this is done, there are still 10 balls, of which 5 are black; $\therefore$ $\frac{5}{10}$ is the probability of drawing a black ball; then Art. 279, the probability of drawing both as required is $\frac{6}{11}\cdot\frac{5}{10}=\frac{3}{11}$.

2. The probability of drawing a white and a black ball in succession is by last Ex. $\frac{3}{11}$; this being done, there are left 5 white and 4 black, and the probability of drawing a black ball from these is $\frac{4}{9}$; $\therefore$ $\frac{3}{11}$ of $\frac{4}{9}=\frac{4}{33}$ is the required probability.

3. The chance of throwing sixes at one throw is $\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{36}$, and of failing $\frac{35}{36}$. Let x be the number of throws required, then $1-(\frac{35}{36})^x=\frac{1}{2}$, (Art. 279), and by the question, x(log. 35 − log. 36) = log. 1 - log. 2, or x(log. 36 − log. 35) = log. 2 $\therefore$ &c.

4. We have here, Art. 280, p. 315, $\dfrac{4.3.2}{1.2.3}\cdot(\frac{1}{6})^3\cdot\frac{5}{6}=\frac{5}{324}$.

5. If two black balls be removed, the 5 remaining balls can be placed in 1.2.3.4.5 positions; and if the 2 balls removed be placed one at each end of the line, we have 2.120 = 240 favourable arrangements; but 2 black balls can be taken out in 3 ways, so that there are in all 6 (1.2.3.4.5) = 1.2.3.4.5.6 favourable arrangements. Now, the 7 balls admit of 1.2.3.4.5.6.7 different arrangements; hence the chance required is $\dfrac{\lfloor 6}{\lfloor 7}=\dfrac{1}{7}$

6. Here, Art. 257 (9), the 10 balls can be drawn 5 together in 252 possible ways; for the favourable cases each two of the 4 white balls may be taken with any three of the

6 black balls; now, 4 balls can be taken 2 together in 6 ways, and 6, three together, in 20 ways. There are $\therefore$ $6.20 = 120$ favourable occurrences; $\therefore$ the chance required is $\frac{120}{252} = \frac{10}{21}$.

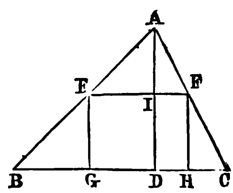
7. The chance of drawing white the first time is $\frac{3}{10}$, and this being done, 9 balls, of which 2 are white, remain, and the chance of drawing white again is $\frac{2}{9}$, and of drawing white the third time $\frac{1}{8}$; hence the chance of drawing white in each of the first three trials is $\frac{3}{10}\cdot\frac{2}{9}\cdot\frac{1}{8} = \frac{1}{120}$.

8. The chance of drawing white is now (Art. 280) $p^3 = (\frac{3}{10})^3 = \frac{27}{1000}$; that of drawing 2 white and 1 black, with or without regard to order, p^2q or $3p^2q$, or $\frac{63}{1000}$ and $\frac{189}{1000}$.

9. The chance of putting the hand in the first urn is $\frac{1}{2}$, and of drawing a white ball therefrom is $\frac{2}{3}$; $\therefore$ the chance of drawing a white ball from the first urn is $\frac{1}{2}$ of $\frac{2}{3} = \frac{1}{3}$; similarly for the second urn, the respective chances are $\frac{1}{2}$, $\frac{4}{5}$ and $\frac{1}{2}$ of $\frac{4}{5} = \frac{2}{5}$, and since the event may happen in either of the two ways, the chance of drawing a white ball from either is $\frac{1}{3} + \frac{2}{5} = \frac{11}{15}$. There being 6 white and 2 black balls, the chance might seem to be $\frac{6}{8}$ in favour of a white ball; but the chances are not the same on account of the number of balls being different. If we had 10 and 5, 12 and 3, then the chance of drawing any of these is equally-probable; $\therefore$ the chance of drawing a white ball is $\frac{22}{30}$ or $\frac{11}{15} = \frac{44}{60}$; but $\frac{6}{8} = \frac{3}{4} = \frac{45}{60}$.

10. After A.'s assertion, $\dfrac{\text{the probability}}{\text{improbability}} = \frac{3}{1}$; B.'s assertion increases the fraction in the ratio of 4 : 1, making it $\frac{12}{1}$; C.'s denial diminishes it in the ratio of 1 : 6; making it $\frac{2}{1}$; hence the probability of the truth : improbability :: 2 : 1.

XXVIII. SOLUTION OF GEOMETRICAL PROBLEMS, p. 324.



1. Let the altitude $AD = a$, the base $BC = b$; and side of square $= x$ $\therefore$ $AI = a - x$; then since the triangles ABC and AEF are similar we have, Euc. VI. 4, BC : EF :: AD : AI; or $a : b :: a - x : x$ $\therefore$ $ax = ab - bx$ and $\therefore$ $x = \dfrac{ab}{a+b}$.

Otherwise, putting $BG = d$, and since (Euc. I., ax. 10) the

area of ABC equals that of the three small triangles and the square, we have $\frac{ab}{2} = \frac{dx}{2} + \frac{(a-x)x}{2} + \frac{x}{2}[b-(d+x)] + x^2$, $\therefore$ $\frac{ab}{2} = \frac{ax}{2} + \frac{bx}{2}$ and $x = \frac{ab}{a+b}$, which is half the harmonic mean between a and b.

2. Here d being the given difference, and putting $AC = x$, fig. 1, p. 321, we have $BC = x + d$; and (Euc. I. 47) $(x+d)^2 = 2x^2$, or $x^2 + 2dx + d^2 = 2x^2$; whence $x^2 - 2dx = d^2$, a quadratic giving $x = d(1 \pm \sqrt{2})$.

3. Let the side $AB = a$, fig. 3, p. 322; put hypotenuse $AC = x$, then $AC + CB = b$ and $CB = b - x$. Therefore (Euc. I. 47) $x^2 = a^2 + (b-x)^2 = a^2 + b^2 - 2bx + x^2$ $\therefore 2bx = a^2 + b^2$, and $x = \frac{a^2+b^2}{2b} = AC$. Also $b - x = b - \frac{a^2+b^2}{2b} = \frac{b^2-a^2}{2b} = BC$.

4. Let the side $= x \therefore$ diagonal $= \sqrt{(2x^2)} = x\sqrt{2}$; then by the question $4x + x\sqrt{2} = s$, or $(4+\sqrt{2})x = s$ $\therefore x = \frac{s}{4+\sqrt{2}}$ Hence, Algebra Art. 117, $x = \frac{4s - s\sqrt{2}}{14}$. Or thus; x being the side we may put diagonal $= s - 4x$; then (Euc. I. 47) $(s-4x)^2 = 2x^2$, whence $14x^2 - 8sx = -s^2$, a quadratic giving $x = \frac{4s \pm s\sqrt{2}}{14}$ = side of square. But diagonal $= x\sqrt{2} = \frac{4s\sqrt{2} - 2s}{14} = \frac{2s\sqrt{2} - s}{7} = \frac{s(2\sqrt{2}-1)}{7}$.

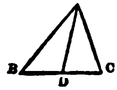
5. Let the sides be x and y; then $x : y :: m : n$; whence $x = \frac{m}{n}.y$. Also (Euc. I. 47, cor.) $x^2 - y^2 = a^2 - b^2$ (see fig. 4, p. 322) that is, $\frac{m^2}{n^2}\cdot y^2 - y^2 = a^2 - b^2$, $\therefore y = n\sqrt{\frac{a^2-b^2}{m^2-n^2}}$; also $\frac{m}{n}y = x = m\sqrt{\frac{a^2-b^2}{m^2-n^2}}$.

6. Here (Ex. 1. p. 320) $x(a-x) = b^2$, and $x^2 - ax = -b^2$, whence $x = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} - b^2\right)}$. The question would be impossible if $b^2 > \frac{a^2}{4}$ or $b > \frac{a}{2}$. Thus the rectangle is a maximum when the line is bisected. (See Euc. II. 5, cor.)

7. Here the segments of the line denoting the width of

the street being x and $a-x$, we have by the question (Euc. I. 47, and ax. 1), $b^2+x^2=(a-x)^2+c^2=a^2-2ax+x^2+c^2$, whence $2ax=a^2-b^2+c^2$ and $x=\frac{a^2-b^2+c^2}{2a}$; $\therefore\ a-x=\frac{a^2+b^2-c^2}{2a}$.

8. Here let AB $=a$, DC $=b$, BD $=x$, AC $=y$; then (Euc. I. 47), A being right, $y^2=(x+b)^2-a^2$. Again (Euc. I. 47, cor.) $a^2-y^2=x^2-b^2$; $\therefore\ a^2-(x+b)^2+a^2=x^2-b^2$, and reducing, $x^2+bx=a^2$; whence $x=\frac{-b\pm\sqrt{(b^2+4a^2)}}{2}$; $\therefore\ b+x=$ hyp. $=b+\frac{-b\pm\sqrt{(b^2+4a^2)}}{2}=\frac{b\pm\sqrt{(b^2+4a^2)}}{2}$. Also $y^2=(x+b)^2-a^2$



$$=\frac{b^2\pm 2b\sqrt{(b^2+4a^2)}+b^2+4a^2}{4}-a^2=$$

$$\frac{b^2\pm 2b\sqrt{(b^2+4a^2)}+b^2+4a^2-4a^2}{4}=\frac{b^2\pm b\sqrt{(b^2+4a^2)}}{2};$$

$\therefore$ side $=y=\sqrt{\frac{b^2\pm b\sqrt{(b^2+4a^2)}}{2}}$.

9. Let the sides be x and y, then $x-y=d$ and (Euc. I. 47) $x^2+y^2=a^2$; but $x^2-2xy+y^2=d^2$; subtracting, $2xy=a^2-d^2$. $\therefore\ 4xy=2(a^2-d^2)$; $\therefore$ by addition, $x^2+2xy+y^2=2a^2-d^2$, and evolving, $x+y=\pm\sqrt{(2a^2-d^2)}$; to obtain x and y combine this with $x-y=d$. Otherwise, the hypotenuse being a, let the base be x; then the perp. will be $x-d$; then (Euc. I. 47) $(x-d)^2+x^2=a^2$; $\therefore\ x^2-2dx+d^2+x^2=a^2$; $\therefore\ x^2-dx=\frac{a^2-d^2}{2}$; $\therefore\ x-\frac{1}{2}d=\pm\sqrt{\left(\frac{d^2}{4}+\frac{a^2-d^2}{2}\right)}=\pm\sqrt{\frac{2a^2-d^2}{4}}$; $\therefore$ $x=\frac{1}{2}d\pm\frac{1}{2}\sqrt{(2a^2-d^2)}=$ base; $\therefore\ x-d=\frac{1}{2}d-d\pm\frac{1}{2}\sqrt{(2a^2-d^2)}$ $=-\frac{1}{2}d\pm\frac{1}{2}\sqrt{(2a^2-d^2)}=$ the perpendicular.

10. Let EAF be the given chord, cutting the diameter BD in A, and let the segments BA, AD of the diameter be b and c; the chord $=a$, and its segments x and $a-x$; then (Euc. III. 35), $x(a-x)=bc$; that is $x^2-ax=-bc$; completing the square and evolving, $x=\frac{a\pm\sqrt{(a^2-4bc)}}{2}$. The position of the chord will be determined by describing a circle from the centre A with radius equal to x, joining A with the point where it meets the given circle, and producing the line to the opposite part of the circumference.

11. Calling x the diagonal, the sides are $x-a$ and $x-b$ by the question. Then (Euc. I. 47) $x^2=(x-a)^2+(x-b)^2=2x^2-2(a+b)x+a^2+b^2$; $\therefore x^2-2(a+b)x=-(a^2+b^2)$. This quadratic gives $x-(a+b)=\pm\sqrt{2ab}$, $\therefore x=a+b\pm\sqrt{2ab}$. Whence the sides $x-a$ and $x-b$ easily. Otherwise, if the sides be called x and y, the diag. $=\sqrt{(x^2+y^2)}$, and by the question $\sqrt{(x^2+y^2)}-x=a$ .. (1); and $\sqrt{(x^2+y^2)}-y=b$ .. (2); the difference of these is $x-y=b-a$.. (3); also $x^2+y^2=a^2+2ax+x^2$; and $y^2=a^2+2ax$.. (4). In (4) put the value of x from (3) and $y^2=a^2+2a(y+b-a)$, whence $y^2-2ay=2ab-a^2$; a quadratic giving $y=a\pm\sqrt{2ab}$. Similarly from (2) and (3) $x^2+y^2=b^2+2by+y^2 \therefore x^2=b^2+2by=b^2+2b(x+a-b)$ $\therefore x^2-2bx=2ab-b^2$, whence $x=b\pm\sqrt{2ab}$. To find the diagonal or $\sqrt{(x^2+y^2)}$ from this, square the values of x and y and add the results; this gets $a^2+b^2+(2a+2b)\sqrt{2ab}+4ab=a^2+2ab+b^2+(2a+2b)\sqrt{2ab}+2ab=\{(a+b)+\sqrt{2ab}\}^2$ $\therefore$ $\sqrt{(x^2+y^2)}=a+b\pm\sqrt{2ab}$.

12. Calling the base x, its segments will be (Alg. p. 25, note) $\frac{1}{2}(x+d)$ and $\frac{1}{2}(x-d)$, then (Euc. II. 5, cor.) $a^2-\left(\frac{x+d}{2}\right)^2=b^2-\left(\frac{x-d}{2}\right)^2$. Whence $dx=a^2-b^2$; and $x=\frac{a^2-b^2}{d}=\frac{(a+b)(a-b)}{d}$. Converting this equation into an analogy, we have $x:a+b=a-b:d$; that is, the base is to the sum of the sides as the difference of the sides is to the difference of the segments. See Thomson's Euc. App. on Trigonometry, prop. V.

13. Put half the base BC $=a$ (fig. 4, p. 322), perp. $=b$; diff. of sides $=d$, and let the intercept $=x$, that is the part of the base between the point of bisection and foot of perp. $\therefore$ the segments of the base made by the perp. are $a+x$ and $a-x$. Hence the sides are $\sqrt{b^2+(a+x)^2}$ and $\sqrt{b^2+(a-x)^2}$, and $\therefore$ by the question $\sqrt{b^2+(a+x)^2}-d=\sqrt{b^2+(a-x)^2}$. Squaring we have $b^2+(a+x)^2-2d\sqrt{b^2+(a+x)^2}+d^2=b^2+(a-x)^2$. Developed and reduced this gives $4ax+d^2=2d\sqrt{b^2+(a+x)^2}$. Squaring again $16a^2x^2+8ad^2x+d^4=4d^2(b^2+a^2+2ax+x^2)$; hence $16a^2x^2+d^4=4d^2(a^2+b^2)+4d^2x^2$, and $\therefore (16a^2-4d^2)x^2=4d^2(a^2+b^2)-d^4$; $\therefore x=\sqrt{\left(\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}\right)}$. This intercept is the answer given in the Algebra. To find the sides,

its value must be substituted in the expressions $\sqrt{b^2+(a+x)^2}$ and $\sqrt{b^2+(a-x)^2}$; this would give

$$\sqrt{\left\{b^2+a^2\pm 2a\sqrt{\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}}+\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}\right\}}.$$

By multiplying b^2+a^2 by the denominator of the last term, and adding the numerators, this reduces to

$$\sqrt{\left\{\pm 2a\sqrt{\frac{4d^2(a^2+b^2)-d^4}{16a^2-4d^2}}+\frac{16a^2(a^2+b^2)-d^4}{16a^2-4d^2}\right\}}$$

in which the upper sign applies to the greater side. Again, let $m:n$ express the ratio of the sides, let a = base, greater segment $=x$, $\therefore$ less $=a-x$, perp. $=b$; then the squares of the sides are b^2+x^2 and $b^2+a^2-2ax+x^2$; hence $m^2:n^2::b^2+x^2:b^2+a^2-2ax+x^2$. Taking the product of extremes and means, $m^2b^2+m^2a^2-2m^2ax+m^2x^2=n^2b^2+n^2x^2$, $\therefore$ $(m^2-n^2)x^2-2m^2ax=(n^2-m^2)b^2-m^2a^2=-(m^2-n^2)b^2-m^2a^2$

or, $$x^2-\frac{2m^2a}{m^2-n^2}.x=-b^2-\frac{m^2a^2}{m^2-n^2}.$$

This quadratic will give the required value for the base segment x, whence $a-x$ and the sides easily. Otherwise, the parts being denoted as above, the sides will be $\sqrt{(x^2+b^2)}$ and $\sqrt{(a-x)^2+b^2}$; and by the question $\sqrt{(x^2+b^2)}-\sqrt{(a-x)^2+b^2}=d$; transposing and squaring $(a-x)^2+b^2=d^2+x^2+b^2-2d\sqrt{(x^2+b^2)}$; whence $a^2-2ax-d^2=-2d\sqrt{(x^2+b^2)}$. Squaring and transposing, $4(a^2-d^2)x^2-4a(a^2-d^2)x=4b^2d^2-(a^2-d^2)^2$; or, $4x^2-4ax=\frac{4b^2d^2}{a^2-d^2}-a^2+d^2$. This quadratic gives for the segment of the base, in terms of the same quantities as before, $x=\frac{a}{2}\pm\frac{d}{2}\sqrt{\frac{4b^2+a^2-d^2}{a^2-d^2}}$. Again we have, $\frac{\sqrt{(x^2+b^2)}}{\sqrt{\{(a-x)^2+b^2\}}}=\frac{m}{n}$; squaring and clearing, $n^2(x^2+b^2)=m^2\{(a-x)^2+b^2\}$; or, $(m^2-n^2)x^2-2m^2ax=-(m^2-n^2)b^2-m^2a^2$, a quadratic which will give the required value for x; whence $a-x$ and the sides.

14. Let (Euc. VI. E.) AD and DC be the chords of the two arcs, and AC the chord of their sum; and let BD $=2r$ be a diameter; put AD $=a$, DC $=b$, AC $=c$, then (Euc. III. 31) the angles A and C will be right, and (Euc. VI. E.) $2r.c=$

aBC + bAB. Also (Euc. I. 47), BC $= \surd(4r^2 - b^2)$ and AB $= \surd(4r^2 - a^2)$, $\therefore\ 2r.c = a\surd(4r^2 - b^2) + b\surd(4r^2 - a^2)$, $\therefore\ c = \frac{1}{2r}\{a\surd(4r^2 - b^2) + b\surd(4r^2 - a^2)\}$, or,

$\frac{a\surd(d^2 - b^2) + b\surd(d^2 - a^2)}{d}$. If the arcs are equal their chords are equal, and the expression becomes,

$$\text{ch. of sum} = \frac{a}{r}\surd(4r^2 - a^2).$$

15. Let BA in fig. of Euc. II. 11 be taken as the radius divided in H, so that AB.BH = AH^2; then AH will be the side of the inscribed decagon (Euc. IV. 11 and cor.); then BA = AC = r, AE = $\frac{1}{2}r$, EB$^2 = \left(\frac{r}{2}\right)^2 + r^2 = \frac{5r^2}{4}$, and EB $= \frac{1}{2}r\surd 5$. Now AH (DE or FC or CG of Euc. IV. 11) = EB − AE $= \frac{1}{2}r\surd 5 - \frac{1}{2}r$; putting the side $= x$, we have $x = \frac{1}{2}r\surd 5 - \frac{1}{2}r = \frac{1}{2}r(\surd 5 - 1)$ for side of decagon inscribed in the circle whose radius $= r =$ BA.

Next, for the side of the inscribed pentagon. From the formula in the last Ex. $c = \frac{a}{r}\surd(4r^2 - a^2)$ we have by substituting $\frac{r}{2}(\surd 5 - 1)$ for a, $\frac{\surd 5 - 1}{2}\surd\{4r^2 - \frac{1}{4}r^2(6 - 2\surd 5)\} =$

$$r.\frac{\surd 5 - 1}{2}\sqrt{\left\{\frac{16 - 6 + 2\surd 5}{4}\right\}} = r.\frac{\surd 5 - 1}{2}\sqrt{\left(\frac{10 + 2\surd 5}{4}\right)}$$

$$= r.\sqrt{\left(\frac{\surd 5 - 1}{2}\right)^2}\sqrt{\left(\frac{10 + 2\surd 5}{4}\right)} =$$

$$r\sqrt{\left\{\frac{(6 - 2\surd 5)(10 + 2\surd 5)}{16}\right\}} = r.\sqrt{\left(\frac{40 - 8\surd 5}{16}\right)} =$$

$r\frac{\surd(40 - 8\surd 5)}{\surd 4.\surd 4} = r\frac{\surd(10 - 2\surd 5)}{\surd 4} = \frac{r}{2}\surd(10 - 2\surd 5) =$ side of inscribed pentagon. It is shewn by writers on Geometry (Thomson's Euc. App. I. 21) that the square of the side of the decagon, together with the square of the radius (side of hexagon), are equal to the square of the side of the pentagon. This gives $x^2 = r^2 + \frac{r^2}{4}(6 - 2\surd 5) = \left(\frac{4r^2}{4} + \frac{6r^2}{4} - \frac{2r^2}{4}\surd 5\right) = \frac{r^2}{4}(10 - 2\surd 5)$, $\therefore x = \frac{r}{2}\surd(10 - 2\surd 5)$, which is the expression for the side of the inscribed pentagon.

Again, for the circumscribed pentagon; denoting by O the middle point of the side FG of the inscribed pentagon (Euc. IV. 11), we have $FO = \frac{1}{2}\ FG = \frac{r}{4}\sqrt{(10-2\sqrt{5})}$, and $DO = \sqrt{(DF^2 - FO^2)} = \sqrt{r^2 - \frac{r^2}{16}(10-2\sqrt{5})} = \sqrt{\{\frac{r^2}{16}(6+2\sqrt{5})\}}$ $= \frac{r}{4}\sqrt{(6+2\sqrt{5})}$. Now a tangent at F, perp. to DF, is a side of the escribed pentagon; let DOC produced meet this side in N (see fig. to prop. XII. and B. Euc. IV.), and let $FN = \frac{1}{2}x$; then by similar triangles NF : FD :: FO : OD, that is, the sides are as the perps. on them from the centre; $\therefore \frac{NF}{r} = \frac{FO}{OD}$, and $NF = \frac{r.FO}{OD}$; that is, $\frac{1}{2}x = r.\frac{\frac{r}{4}\sqrt{(10-2\sqrt{5})}}{\frac{r}{4}\sqrt{(6+2\sqrt{5})}}$ $= r.\frac{\sqrt{(10-2\sqrt{5})}}{\sqrt{(6+2\sqrt{5})}}$; to rationalize the denominator, multiply both terms by $\sqrt{(6-2\sqrt{5})}$; then $\frac{1}{2}x = r\sqrt{\frac{80-32\sqrt{5}}{16}} = r\sqrt{(5-2\sqrt{5})}$. Hence side of escribed pentagon, $x = 2r\sqrt{(5-2\sqrt{5})}$.

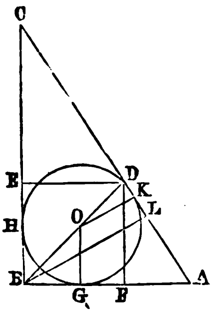
Lastly, for side of escribed decagon, if from B and F (Thomson's Euc. App. I. 21) tangents be drawn meeting in a point, the two lines thus formed are each half the side of the escribed decagon, *i.e.*, $\frac{1}{2}x$. If we call the point of meeting L, the triangles LBF and FBC are similar, each being isosceles, and having the angle at the vertex the angle of a decagon; we have thus (same fig.) $\frac{1}{2}x : FB :: FB : BC$ and $\therefore \frac{\frac{1}{2}x}{FB} = \frac{FB}{BC}$ or $\frac{\frac{1}{2}x}{\frac{r}{2}(\sqrt{5}-1)} = \frac{\frac{r}{2}(\sqrt{5}-1)}{\frac{r}{2}\sqrt{(10-2\sqrt{5})}} = \frac{\sqrt{5}-1}{\sqrt{(10-2\sqrt{5})}}$; divide now both terms by $\sqrt{(\sqrt{5}-1)}$ and we obtain $\frac{\sqrt{(\sqrt{5}-1)}}{\sqrt{(2\sqrt{5})}}$ since $10-2\sqrt{5} = (\sqrt{5}-1)(2\sqrt{5})$. Hence $\frac{1}{2}x = \frac{r}{2}.\frac{\sqrt{(\sqrt{5}-1)^2(\sqrt{5}-1)}}{\sqrt{(2\sqrt{5})}}$

$$= \frac{r}{2}.\frac{\sqrt{(\sqrt{5}-1)^3.2\sqrt{5}}}{2\sqrt{5}} = r.\frac{\sqrt{(80-32\sqrt{5})}}{2.2\sqrt{5}} =$$

$$r.\frac{\sqrt{16}\sqrt{(5-2\sqrt{5})}}{4\sqrt{5}} = r.\frac{\sqrt{(5-2\sqrt{5})}}{\sqrt{5}}$$

$$= r.\sqrt{\frac{5-2\sqrt{5}}{5}}.\quad \text{Hence side} = 2.r\sqrt{\frac{5-2\sqrt{5}}{5}}.$$

16. Let A B C be the given triangle, B right; then the diagonal B D of the square passes through the centre O of the inscribed circle, whose radius (Euc. IV. 4, cor. 5) is B G or B H, and bisects the right angle B. Draw the radii O G, O K to the points of contact, and let fall B L perp. to the hypotenuse. Let now radius $= b$, side of square $= a$, segment $\text{A L} = x$; then, by virtue of the parallels, BF : FG :: BD : DO :: BL : OK; or, $a : a - b ::$ BL $: b$; $\therefore$ B L$=\dfrac{ab}{a-b}$; whence $\text{D L} = \sqrt{(\text{B D}^2 - \text{B L}^2)}$ since L is a rightangle; *i.e.*, $\text{D L} = \sqrt{\left(2a^2 - \dfrac{a^2b^2}{(a-b)^2}\right)}$



Put this $= c$, and $\dfrac{ab}{a-b} = d$. Then (Euc. VI. 3, and 8 cor.) A L : L B :: L B : L C, and A D : D C .: A B : B C :: A L : L B; *i.e.*, $x : d :: d :$ C L $= \dfrac{d^2}{x}$, $\therefore$ C D $= \dfrac{d^2}{x} - c$; also $x + c : \dfrac{d^2}{x} - c :: x : d$; hence, taking the rectangle of extremes and means $dx + cd = d^2 - cx$. Whence $x = \dfrac{d(d-c)}{d+c}$. This will determine the triangle; for the position of D being given, the tangent through it is the hypotenuse; and AL being found, the preceding proportions will give BL and LC in absolute length; also BD is known, being the diagonal of a given square.

APPENDIX.

1. If s_1, s_2, s_3, &c., be the sum to n terms of n Geometric series, whose first terms are each unity and common ratios, 1, 2, 3, &c.: prove that

$$s_1 + s_2 + 2s_3 + 3s_4 + \quad . \quad . \quad . \quad (n-1)s_n = 1^n + 2^n + 3^n + . + n^n.$$

$$s_1 = 1 + 1 + \ 1 + \ . \ . \ . \ . \ . \ . \ . \ . \ + 1 \quad = n$$
$$s_2 = 1 + 2 + \ 4 + \ . \ . \ . \ . \ . \ . \ . \ . \ + 2^{n-1} = 2^n - 1$$
$$s_3 = 1 + 3 + \ 9 + \ . \ . \ . \ . \ . \ . \ . \ . \ + 3^{n-1} = \frac{3^n - 1}{2}$$
$$s_4 = 1 + 4 + 16 + \ . \ . \ . \ . \ . \ . \ . \ . \ + 4^{n-1} = \frac{4^n - 1}{3}$$
$$\vdots$$
$$s_n = 1 + n + n^2 + \ . \ . \ . \ . \ . \ . \ . \ . \ + n^{n-1} = \frac{n^n - 1}{n-1}$$

or, $s_1 = n$, $s_2 = 2^n - 1$, $2s_3 = 3^n - 1$, $3s_4 = 4^n - 1$

$$\vdots$$

$$(n-1)s_n = n^n - 1$$

$$\therefore s_1 + s_2 + 2s_3 + 3s_4 + \ldots + (n-1)s_n = 2^n + 3^n + 4^n + \text{\&c.} + n - (n-1)$$
$$= 1^n + 2^n + 3^n + 4^n \ . \ . \ . \ + n^n.$$

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2. If s_1, s_2, s_3, &c., are the sums of m Arithmetic series each to n terms, the first terms being 1, 2, 3, &c., and the differences 1, 3, 5, &c.: prove that

$$s_1 + s_2 + s_3 + \text{\&c.} = \tfrac{1}{2}mn(mn + 1).$$

$$s_1 = 1 + \ 2 + \ 3 + \ 4 + \ . \ . \ . \ . \ + n \qquad = \frac{n}{2}\{\ n + 1\}$$
$$s_2 = 2 + \ 5 + \ 8 + 11 + \ . \ . \ . \ . \ + (3n - 1) = \frac{n}{2}\{3n + 1\}$$
$$s_3 = 3 + \ 8 + 13 + 18 + \ . \ . \ . \ . \ + (5n - 2) = \frac{n}{2}\{5n + 1\}$$

$$s_4 = 4 + 11 + 18 + 25 + \ldots + (7n-3) = \frac{n}{2}\{7n+1\}$$

$$\vdots$$

$s_m = m + (3m-1) + (5m-2) + \ldots + \{m + (n-1)(2m-1)\}$ $= \frac{n}{2}\{(2m-1)n+1\}$ $\therefore s_1 + s_2 + s_3 + \ldots + s_m = \frac{n}{2}\{n + 3n + 5n + \ldots + (2m-1)n\} + \frac{mn}{2} = \frac{n^2}{2}\{1 + 3 + 5 + \ldots + (2m-1)\} + \frac{mn}{2} = \frac{n^2}{2} \times m^2 + \frac{mn}{2} = \frac{m^2n^2}{2} + \frac{mn}{2} = \frac{mn}{2}\{mn+1\}.$

3. If P, Q, and R denote respectively the pth, qth, and rth terms of an arithmetic progression, prove the following relation: $(q-r)\,P + (r-p)\,Q + (p-q)\,R = 0.$

We have $a + (p-1)d = P \quad \ldots \quad (1)$

$a + (q-1)d = Q \quad \ldots \quad (2) \qquad a + (r-1)d = R \quad \ldots \quad (3)$

Subtracting (2) from (1) and (3) from (2),

$$\therefore \frac{P - Q}{Q - R} = \frac{p-q}{q-r};$$

$$\therefore (q-r)\,P + (r-p)\,Q + (p-q)\,R = 0.$$

4. If P, Q, and R denote respectively the pth, qth, and rth terms of a geometrie progression, prove the following relation:

$$P^{q-r}\,Q^{r-p}\,R^{p-q} = 1.$$

If a denotes the first term of the progression and k the ratio, we have

$$ak^{p-1} = P \quad \ldots \quad (1)$$
$$ak^{q-1} = Q \quad \ldots \quad (2)$$
$$ak^{r-1} = R \quad \ldots \quad (3)$$

Dividing (1) by (2) $\therefore \frac{P}{Q} = k^{p-q} \quad \therefore k = \left(\frac{P}{Q}\right)^{\frac{1}{p-q}}$

(2) by (3) $\therefore \frac{Q}{R} = k^{q-r} \quad \therefore k = \left(\frac{Q}{R}\right)^{\frac{1}{q-r}}$

$$\therefore \left(\frac{P}{Q}\right)^{q-r} = \left(\frac{Q}{R}\right)^{p-q} \therefore P^{q-r}\,R^{p-q} = Q^{p-r}$$

Multiply both sides by Q^{r-p}

$$\therefore P^{q-r}\,Q^{r-p}\,R^{p-q} = 1.$$

5. Shew that in the series 1, 3, 5, &c., the sum of the first half of any even number of terms bears a constant ratio to the sum of the last half.

Let $2p$ denote the number of terms; prove that the sum of the first p terms bears a constant ratio to the sum of the last p terms.

Let $s_1 = 1 + 3 + 5 + \ldots + (2p-1) = p^2.$

$s_2 = (2p+1) + (2p+3) + (2p+5) \ldots + (4p-1) = 3p^2.$

$$\therefore \frac{s_1}{s_2} = \frac{p^2}{3p^2} = \frac{1}{3} = \text{a constant.}$$

6. s is the sum of three terms in a Geometric progression, of which the first is unity, and s is the sum of the reciprocals of the same numbers, prove that the sum *ad infinitum* is $s \div \left\{1 - \left(\frac{s}{s'}\right)^{\frac{3}{2}}\right\}$.

Let r denote the ratio, and Σ the sum *ad infinitum*;

$$\text{we have } s = 1 + r + r^2 = \frac{1}{1-r}(1-r^3) = \Sigma(1-r^3), \ldots (1)$$

$$\text{and } s' = 1 + \frac{1}{r} + \frac{1}{r^2} = \frac{1}{r^2}(1 + r + r^2) = \frac{1}{r^2}s \ldots (2)$$

$$\text{From (1) we have } \Sigma = \frac{s}{1-r^3} \ldots (3)$$

$$\text{From (2) we have } r^2 = \frac{s}{s'} \therefore r = \left(\frac{s}{s'}\right)^{\frac{1}{2}}$$

Substituting this value of r in (3), we have $\Sigma = \dfrac{s}{1 - \left(\frac{s}{s'}\right)^{\frac{3}{2}}}$.

7. If n Arithmetic and Geometric means be inserted between a and l, find the mth mean in each case.

For the Arithmetic means we have $l = a + (n+1)d \therefore d = \dfrac{l-a}{n+1}$, which is the common difference.

Now the mth mean is evidently the $(m+1)$th term, reckoning from beginning, and is therefore $= a + md = a + \dfrac{m(l-a)}{n+1} = \dfrac{a(n-m+1)+ml}{n+1}$.

For the Geometric means we have $l = ar^{n+1} \therefore r = \left(\dfrac{l}{a}\right)^{\frac{1}{n+1}}$ which is the common ratio.

As before, the mth mean is the $(m+1)$th term, reckon-

ing from beginning, and is evidently $=ar^m = a\left(\frac{l}{a}\right)^{\frac{m}{n+1}} =$ $\sqrt[n+1]{l^m.a^{1+n-m}}$

8. If P be the continued product of n quantities in Geo metric progression, s their sum, and s_1 the sum of their reciprocals, shew that

$$P^2 = \left(\frac{s}{s_1}\right)^n$$

Let a, ar, ar^2, ar^3 . . . . ar^{n-1} be the n quantities in Geometric progression,

Then $P = a \times ar \times ar^2 \times \ . \ . \ . \ . \times ar^{n-1}$

$= a^n r^{1+2+3+ \ . \ . \ . \ . \ +(n-1)}$

$= a^n r^{\frac{n(n-1)}{2}}$

And $s = a\{1 + r + r^2 + \ . \ . \ . \ + r^{n-1}\} \therefore 1 + r + r^2 + \&c. = \frac{s}{a}$

$$s_1 = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \ \&c., \ = \frac{1}{ar^{n-1}}\{1 + r + r^2 + \&c.\}$$

$$\therefore s_1 = \frac{1}{ar^{n-1}} \times \frac{s}{a}$$

$$\therefore a^2 r^{n-1} = \frac{s}{s_1}$$

$$\therefore a^{2n} r^{n(n-1)} = \left(\frac{s}{s_1}\right)^n$$

$$\text{But } P = a^n r^{\frac{n(n-1)}{2}}$$

$$\therefore P^2 = a^{2n} r^{n(n-1)} = \left(\frac{s}{s_1}\right)^n.$$

9. Find the ratio of the latter half of $2n$ terms of an arithmetic progression, to the sum of $3n$ terms of the same series.

Let a = first term

b = common difference,

the series will be

$$a + \overline{a + b} + \overline{a + 2b} + \ . \ . \ . \ . \ \&c.,$$

the last half of $2n$ terms of this series will begin with the $(n + 1)$th term, and its sum is evidently equal to

$$\frac{n}{2}\{2a + 2nb + (n - 1)b\} = \frac{n}{2}\{2a + 3nb - b\}$$

The sum of $3n$ terms of the same series is

$\frac{3n}{2}\{2a+3nb-b\}$ $\therefore$ $\frac{\text{sum of latter half of } 2n \text{ terms}}{\text{sum of } 3n \text{ terms}} =$

$$\frac{\frac{n}{2}\{2a+3nb-b\}}{\frac{3n}{2}\{2a+3nb-b\}} = \frac{1}{3}$$

10. The nth term of an arithmetic series is $\frac{1}{6}(3n-1)$, find the first term, common difference, sum of n terms:—

Let $n=1$, $\therefore$ the first term $=\frac{1}{3}$. Let $n=2$, $\therefore$ second term $=\frac{5}{6}$; $\therefore$ common difference $=\frac{1}{2}$. Sum of n terms $=\frac{n}{2}\{a+l\}$ $=\frac{n}{2}\left\{\frac{1}{3}+\frac{1}{6}(3n-1)\right\} = \frac{n}{2}\left\{\frac{1}{6}+\frac{n}{2}\right\} = \frac{n}{12}\{3n+1\}$.

11. If s_n, s_{n+1}, s_{n+2}, s_{n+3}, &c., denote respectively the sum of n, $(n+1)$, $(n+2)$, $(n+3)$, &c., terms of the arithmetic progression, whose first term is a, and common difference b; find $s_n+s_{n+1}+s_{n+2}+s_{n+3}+$ &c., to n terms.

$$\begin{aligned} \text{We have } 2s_n &= n\{2a+(n-1)b\} \\ 2s_{n+1} &= (n+1)\{2a+n\cdot b\} \\ 2s_{n+2} &= (n+2)\{2a+(n+1)b\} \\ &\vdots \\ \&c. &= \;\; \&c. \\ &\vdots \end{aligned}$$

$\therefore$ $2s_n+2s_{n+1}+2s_{n+2}+$ &c. $=\{n+(n+1)+(n+2)+(n+3)+$ &c. to n terms$\}\,2a+\{(n-1)n+n(n+1)+(n+1)(n+2)+$ &c. to n terms$\}b$.

The last part of the right hand member is evidently $=$ $\{n^2-n+(n+1)^2-(n+1)+(n+2)^2-(n+2)+\text{\&c.}\}b=\{n^2+(n+1)^2+(n+2)^2+(n+3)^2+(n+4)^2+\text{\&c.} \;.\;.\;.\;\}b-\{n+(n+1)+(n+2)+(n+3)+\text{\&c.}\}b$, $\therefore$ $2s_n+2s_{n+1}+2s_{n+2}+\text{\&c.}=(2a-b)\{n+n+1+(n+2)+(n+3)+\text{\&c.}\}+\{n^2+(n+1)^2+(n+2)^2+(n+3)^2+\text{\&c}\}b=(2a-b)\times\frac{n}{2}\{3n-1\}+\frac{1}{6}n(2n-1)(7n-1)$.

[For the sum of the series $n^2 + (n+1)^2 + (n+2)^2 +$ &c., see Text Book, p. 258, where it is shewn that, $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{1.2.3.}$; $\therefore 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2 +$ &c. to $2n - 1$ terms $= \frac{(2n-1)(2n)(4n-1)}{1.2.3.}$, by substituting $2n - 1$ for n. But $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots (n-1)^2 = \frac{(n-1)n(2n-1)}{1.2.3.}$, $\therefore n^2 + (n+1)^2 + (n+2)^2 +$ &c. $= \frac{1}{6}n(2n-1)(7n-1)$].

FINIS.

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